

SHARP CUSA TYPE INEQUALITIES FOR TRIGONOMETRIC FUNCTIONS WITH TWO PARAMETERS

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ABSTRACT. Let $(p, q) \mapsto \beta(p, q)$ be a function defined on \mathbb{R}^2 . We determine the best or better p, q such that the inequality

$$\left(\frac{\sin x}{x}\right)^p < (>) 1 - \beta(p, q) + \beta(p, q) \cos^q x$$

holds for $x \in (0, \pi/2)$, and obtain a lot of new and sharp Cusa type inequalities for trigonometric functions. As applications, some new Shafer-Fink type and Carlson type inequalities for arc sine and arc cosine functions, and new inequalities for trigonometric means are established.

1. INTRODUCTION

For $x \in (0, \pi/2)$, the double inequality

$$(1.1) \quad (\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}$$

holds true, where the left inequality was obtained by Adamović and Mitrinović (see [1, 2, p. 238]), while the right one is due to Cusa and Huygens (see, e.g., [2]) and it is now known as *Cusa's inequality* [3], [4], [5], [6], [7].

There are many improvements, refinements and generalizations of (1.1). For the first inequality in (1.1), a nice refinement has appeared in [1, 3.4.6], which states that for $x \in (0, \pi/2)$, the inequalities

$$(\cos x)^{1/3} < \cos \frac{x}{\sqrt{3}} < \frac{\sin x}{x}$$

hold. Neuman presented an interesting chain of inequalities in [8, Theorem 1] (also see [5], [9], [10], [11, (3.23)]), that is, the inequalities

$$(1.2) \quad \begin{aligned} (\cos x)^{1/3} &< \left(\cos x \frac{\sin x}{x}\right)^{1/4} < \left(\frac{\sin x}{\operatorname{arctanh} \sin x}\right)^{1/2} < \left(\frac{\cos x + (\sin x)/x}{2}\right)^{1/2} \\ &< \left(\frac{1 + 2 \cos x}{3}\right)^{1/2} < \left(\frac{1 + \cos x}{2}\right)^{2/3} < \frac{\sin x}{x} \end{aligned}$$

are valid for $x \in (0, \pi/2)$.

For the second one in (1.1), Yang [12] and Klén et al. [13, Theorem 2.4] showed that for $x \in (0, \pi)$

$$(1.3) \quad \frac{\sin x}{x} \leq \cos^3 \frac{x}{3} \leq \frac{2 + \cos x}{3}.$$

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Further, Yang [14] has shown that for $x \in (0, \pi/2)$ the inequalities

$$(1.4) \quad \frac{\sin x}{x} < \left(\frac{2}{3} \cos \frac{x}{2} + \frac{1}{3}\right)^2 < \cos^3 \frac{x}{3} < \frac{2 + \cos x}{3}$$

are true.

By constructing a monotonic function $p \mapsto (\cos px)^{1/(3p^2)}$ ($p \in (0, 1]$), Yang [15] showed that the inequalities

$$(1.5) \quad (\cos x)^{1/3} < \cos \frac{x}{\sqrt{3}} < \left(\cos \frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} \\ < \left(\cos \frac{x}{3}\right)^3 < \left(\cos \frac{x}{4}\right)^{16/3} < e^{-x^2/6} < \frac{2 + \cos x}{3}$$

are valid for $x \in (0, \pi/2)$.

It is worth mentioning that Zhu [7] established a more general result containing Cusa-type inequalities, which is recorded as follows.

Theorem Zhu ([7]) *Let $0 < x < \pi/2$. Then*

(i) *if $p \geq 1$, the double inequality*

$$(1.6) \quad 1 - \xi + \xi (\cos x)^p < \left(\frac{\sin x}{x}\right)^p < 1 - \eta + \eta (\cos x)^p$$

holds if and only if $\eta \leq 1/3$ and $\xi \geq 1 - (2/\pi)^p$;

(ii) *if $0 \leq p \leq 4/5$, the double inequality*

$$(1.7) \quad 1 - \eta + \eta (\cos x)^p < \left(\frac{\sin x}{x}\right)^p < 1 - \xi + \xi (\cos x)^p$$

holds if and only if $\eta \geq 1/3$ and $\xi \leq 1 - (2/\pi)^p$,

(iii) *if $p < 0$, the inequality*

$$\left(\frac{\sin x}{x}\right)^p < 1 - \eta + \eta (\cos x)^p$$

holds if and only if $\eta \geq 1/3$.

As a consequence of Theorem Zhu, the double inequality

$$\left(\frac{2}{3} + \frac{1}{3} (\cos x)^{4/5}\right)^{5/4} < \frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3} \cos x$$

holds $0 < x < \pi/2$, where $4/5$ is the best. He [16] and Yang [17] showed independently that the double inequality

$$(1.8) \quad \left(\frac{2}{3} + \frac{1}{3} (\cos x)^{4/5}\right)^{5/4} < \frac{\sin x}{x} < \left(\frac{2}{3} + \frac{1}{3} (\cos x)^{p_0^*}\right)^{1/p_0^*},$$

holds for $x \in (0, \pi/2)$ with the best constants $4/5$ and $p_0^* = \log_{\pi/2}(3/2) \approx 0.8979$.

The aim of this paper is to determine the best or better p, q such that the inequality

$$(1.9) \quad \left(\frac{\sin x}{x}\right)^p < (>) 1 - \beta + \beta \cos^q x$$

or

$$(1.10) \quad \frac{\sin x}{x} < (>) (1 - \beta + \beta \cos^q x)^{1/p}$$

holds for $x \in (0, \pi/2)$.

The paper is organized as follows. In Section 2, we investigate the monotonicity of the function $T_{p,q}$ defined on $(0, \pi/2)$ by

$$(1.11) \quad T_{p,q}(x) = \frac{U_p\left(\frac{\sin x}{x}\right)}{U_q(\cos x)},$$

where $p, q \in \mathbb{R}$ and U_p is defined on $(0, 1)$ by

$$(1.12) \quad U_p(t) = \frac{1-t^p}{p} \text{ if } p \neq 0 \text{ and } U_0(t) = -\ln t.$$

In Section 3, by using the monotonicity of $T_{p,q}$ on $(0, \pi/2)$, we prove some sharp Cusa type inequalities for trigonometric functions for certain p, q . It is not only to generalize Zhu's results, but also present many new and interesting inequalities for trigonometric functions. In the last section, as applications, some new inequalities for arc sine function and bivariate means are presented.

2. MONOTONICITY

We begin with the following simple assertion.

Lemma 1. *Let the function U_p defined on $(0, 1)$ by (1.12). Then $p \mapsto U_p(t)$ is decreasing on \mathbb{R} and $U_p(t) > 0$ for $t \in (0, 1)$.*

Proof. For $p \neq 0$, differentiation yields

$$\frac{\partial}{\partial p} U_p(t) = \frac{1}{p^2} (t^p - 1) - \frac{1}{p} t^p \ln t = \frac{t^p}{p^2} (\ln t^{-p} - (t^{-p} - 1)) < 0,$$

where the last inequality holds due to $\ln x \leq (x - 1)$ for $x > 0$.

Employing the decreasing property, we get $U_p(t) > \lim_{p \rightarrow \infty} U_p(t) = 0$, which proves the lemma. \square

For $x \in (0, \pi/2)$, we denote by

$$S_p(x) := U_p\left(\frac{\sin x}{x}\right) \quad \text{and} \quad C_p(x) := U_p(\cos x)$$

due to $(\sin x)/x, \cos x \in (0, 1)$. Then we have

$$(2.1) \quad S_p(x) = \frac{1 - \left(\frac{\sin x}{x}\right)^p}{p} \text{ if } p \neq 0 \quad \text{and} \quad S_0(x, p) = -\ln \frac{\sin x}{x} \text{ if } p = 0,$$

$$(2.2) \quad C_p(x) = \frac{1 - \cos^p x}{p} \text{ if } p \neq 0 \quad \text{and} \quad C_0(x, p) = -\ln(\cos x) \text{ if } p = 0.$$

And then, the function $x \mapsto T_{p,q}(x) = U_p\left(\frac{\sin x}{x}\right)/U_q(\cos x) = S_p(x)/C_q(x)$ can be expressed as

$$(2.3) \quad T_{p,q}(x) = \begin{cases} \frac{q}{p} \frac{1 - \left(\frac{\sin x}{x}\right)^p}{1 - \cos^q x} & \text{if } pq \neq 0, \\ \frac{1}{p} \frac{\left(\frac{\sin x}{x}\right)^p - 1}{\ln(\cos x)} & \text{if } p \neq 0, q = 0, \\ q \frac{\ln \frac{\sin x}{x}}{\cos^q x - 1} & \text{if } p = 0, q \neq 0, \\ \frac{\ln \frac{\sin x}{x}}{\ln(\cos x)} & \text{if } p = q = 0. \end{cases}$$

In order to investigate the monotonicity of the function $T_{p,q}$, we first recall the following important lemmas.

Lemma 2 ([19], [20]). *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If f'/g' is increasing (or decreasing) on (a, b) , then so are the functions*

$$x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)}.$$

The following lemma is crucial to prove certain best inequalities, which is inspired by part (iv) of proof of Theorem 6 in [23].

Lemma 3. *Suppose that $f, g : [a, b] \rightarrow \mathbb{R}$ are two continuous functions which are differentiable on (a, b) and $g' \neq 0$ on (a, b) . If f'/g' is increasing (decreasing) on (a, x_0) and decreasing (increasing) on (x_0, b) , and*

$$(2.4) \quad \frac{f(b) - f(a)}{g(b) - g(a)} \geq (\leq) \frac{f'(a^+)}{g'(a^+)} = \lambda \neq \pm\infty,$$

then the inequality

$$(2.5) \quad \frac{f(x) - f(a)}{g(x) - g(a)} > (<) \lambda$$

holds for all $x \in (a, b)$.

Proof. Without loss of generality, we assume that $g' > 0$ on (a, b) .

For $x \in (a, x_0)$, by Lemma 2, since f'/g' is increasing (decreasing) on (a, x_0) , so is the function

$$x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)}.$$

Then we get that for $x \in (a, x_0]$

$$(2.6) \quad \frac{f(x) - f(a)}{g(x) - g(a)} > (<) \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(a^+)}{g'(a^+)} = \lambda.$$

That is, the inequality (2.5) holds for $x \in (a, x_0]$.

On the other hand, from Lemma 2, that f'/g' is decreasing (increasing) on (x_0, b) means that so is the function

$$x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)},$$

and hence we have

$$\frac{f(x) - f(b)}{g(x) - g(b)} < (>) \frac{f(x_0) - f(b)}{g(x_0) - g(b)} \text{ for } x \in (x_0, b),$$

which can be rewritten as

$$f(x) > (<) f(b) + \frac{f(x_0) - f(b)}{g(x_0) - g(b)} (g(x) - g(b)) := \phi(x),$$

due to assumption that $g' > 0$ on (a, b) . Clearly, in order to prove the desired inequality, it suffices to prove

$$\phi(x) > (<) f(a) + \lambda (g(x) - g(a)) \text{ for } x \in (x_0, b).$$

Since x_0 satisfies the relation (2.6), that is,

$$f(x_0) > (<) f(a) + \lambda (g(x_0) - g(a)),$$

which together with (2.4), that is,

$$f(b) \geq (\leq) f(a) + \lambda (g(b) - g(a)),$$

leads to

$$\begin{aligned}
\phi(x) &= \frac{g(x) - g(b)}{g(x_0) - g(b)} f(x_0) + \frac{g(x_0) - g(x)}{g(x_0) - g(b)} f(b) \\
&> (<) \frac{g(x) - g(b)}{g(x_0) - g(b)} (f(a) + \lambda(g(x_0) - g(a))) \\
&\quad + \frac{g(x_0) - g(x)}{g(x_0) - g(b)} (f(a) + \lambda(g(b) - g(a))) \\
&= f(a) + \lambda(g(x) - g(a)).
\end{aligned}$$

This means that the inequality (2.5) also holds for $x \in (x_0, b)$. Thus the proof is completed. \square

Lemma 4 ([21]). *Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \dots$, and a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 5 ([22, pp.227-229]). *We have*

$$(2.7) \quad \frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \pi$$

$$(2.8) \quad \cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \pi,$$

$$(2.9) \quad \frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| x^{2n-2}, \quad |x| < \pi,$$

$$(2.10) \quad \tan x = \sum_{n=1}^{\infty} \frac{2^{2n} - 1}{(2n)!} 2^{2n} |B_{2n}| x^{2n-1}, \quad |x| < \pi/2.$$

where B_n is the Bernoulli number.

Now we are in position to prove the monotonicity of $T_{p,q}$. Clearly, $T_{p,q}(x)$ can be written as

$$T_{p,q}(x) = \frac{S_p(x)}{C_q(x)} = \frac{S_p(x) - S_p(0^+)}{C_q(x) - C_q(0^+)}.$$

For $pq \neq 0$, differentiation yields

$$(2.11) \quad \frac{S'_p(x)}{C'_q(x)} = \frac{\cos^{1-q} x}{x^2 \sin x} \left(\frac{\sin x}{x} \right)^{p-1} (\sin x - x \cos x) := f_1(x)$$

$$(2.12) \quad f'_1(x) = -\frac{1}{x^2 \sin^3 x \cos^q x} \left(\frac{\sin x}{x} \right)^p \times f_2(x),$$

where

$$(2.13) \quad f_2(x) = pA(x) - qB(x) + C(x),$$

in which

$$(2.14a) \quad A(x) = (\sin x - x \cos x)^2 \cos x > 0,$$

$$(2.14b) \quad B(x) = x(\sin x - x \cos x) \sin^2 x > 0$$

$$(2.14c) \quad C(x) = -(2x^2 \cos x - x \sin x - \cos x \sin^2 x) > 0,$$

here $C(x) > 0$ due to

$$C(x) = x^2 (\cos x) \left(\frac{\sin^2 x}{x^2} + \frac{\tan x}{x} - 2 \right) > 0$$

by Wilker inequality (see [18]). It is easy to verify that (2.11), (2.12) and (2.13) are true for $pq = 0$.

We see clearly that, by Lemma 2, if we can prove $f_2(x) \leq (\geq) 0$ for all $x \in (0, \pi/2)$ then $T_{p,q}$ defined by (2.3) is increasing (decreasing) on $(0, \pi/2)$. In order to prove it, we need the expansions of $A(x)$, $B(x)$ and $C(x)$. Using Lemma 5 we get

$$\begin{aligned} \frac{A(x)}{\sin^2 x \cos x} &= \frac{x^2 \cos^3 x - 2x \cos^2 x \sin x + \cos x \sin^2 x}{\sin^2 x \cos x} = x^2 \frac{1}{\sin^2 x} - x^2 - 2x \frac{\cos x}{\sin x} + 1 \\ &= x^2 \left(\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| x^{2n-2} \right) - x^2 \\ &\quad - 2x \left(\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1} \right) + 1 \\ (2.15) \quad &= \sum_{n=2}^{\infty} \frac{(2n+1) 2^{2n}}{(2n)!} |B_{2n}| x^{2n} := \sum_{n=2}^{\infty} a_n x^{2n}, \end{aligned}$$

$$\begin{aligned} \frac{B(x)}{\sin^2 x \cos x} &= \frac{x \sin^3 x - x^2 \cos x \sin^2 x}{\sin^2 x \cos x} = x \frac{\sin x}{\cos x} - x^2 \\ &= \sum_{n=1}^{\infty} \frac{2^{2n} - 1}{(2n)!} 2^{2n} |B_{2n}| x^{2n} - x^2 \\ (2.16) \quad &= \sum_{n=2}^{\infty} \frac{(2^{2n} - 1) 2^{2n}}{(2n)!} |B_{2n}| x^{2n} := \sum_{n=2}^{\infty} b_n x^{2n}, \end{aligned}$$

$$\begin{aligned} \frac{C(x)}{\sin^2 x \cos x} &= \frac{-2x^2 \cos x + x \sin x + \cos x \sin^2 x}{\sin^2 x \cos x} = -2x^2 \frac{1}{\sin^2 x} + 2x \frac{1}{\sin 2x} + 1 \\ &= -2x^2 \left(\frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| x^{2n-2} \right) \\ &\quad + 2x \left(\frac{1}{2x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| 2^{2n-1} x^{2n-1} \right) + 1 \\ (2.17) \quad &= \sum_{n=2}^{\infty} \frac{(2^{2n} - 4n) 2^{2n}}{(2n)!} |B_{2n}| x^{2n} := \sum_{n=2}^{\infty} c_n x^{2n}. \end{aligned}$$

Lemma 6. Let g_1 be defined on $(0, \pi/2)$ by

$$(2.18) \quad g_1(x) = \frac{qB(x) - C(x)}{A(x)}.$$

where $A(x)$, $B(x)$ and $C(x)$ are defined by (2.14a), (2.14b) and (2.14c), respectively. Then

(i) g_1 is increasing on $(0, \pi/2)$ if $q \geq 1$, and we have

$$(2.19) \quad 3q - \frac{8}{5} < g_1(x) < \begin{cases} \infty & \text{if } q > 1, \\ \frac{\pi^2}{4} - 1 & \text{if } q = 1; \end{cases}$$

(ii) g_1 is decreasing on $(0, \pi/2)$ if $q \leq 34/35$, and we have

$$(2.20) \quad -\infty < g_1(x) < 3q - \frac{8}{5}.$$

Proof. Using (2.15), (2.16) and (2.17) we have

$$g_1(x) = \frac{qB(x) - C(x)}{A(x)} = \frac{\sum_{n=2}^{\infty} (qb_n - c_n)x^{2n}}{\sum_{n=2}^{\infty} a_n x^{2n}} \quad \text{and} \quad \frac{qb_n - c_n}{a_n} = \frac{(2^{2n} - 1)q - (2^{2n} - 4n)}{(2n+1)}.$$

And then,

$$\begin{aligned} \frac{qb_{n+1} - c_{n+1}}{a_{n+1}} - \frac{qb_n - c_n}{a_n} &= \frac{(2^{2(n+1)} - 1)q - (2^{2(n+1)} - 4(n+1))}{(2(n+1)+1)} - \frac{(2^{2n} - 1)q - (2^{2n} - 4n)}{(2n+1)} \\ &= \frac{(6n+1)4^n + 2}{(2n+3)(2n+1)} \left(q - 1 + \frac{6}{(6n+1)4^n + 2} \right). \end{aligned}$$

From this it is obtained that

$$\frac{qb_{n+1} - c_{n+1}}{a_{n+1}} - \frac{qb_n - c_n}{a_n} \begin{cases} > 0 & \text{if } q \geq \sup_{n \in \mathbb{N}, n \geq 2} \left(1 - \frac{6}{(6n+1)4^n + 2} \right) = 1, \\ < 0 & \text{if } q \leq \inf_{n \in \mathbb{N}, n \geq 2} \left(1 - \frac{6}{(6n+1)4^n + 2} \right) = \frac{34}{35}. \end{cases}$$

In the case of $q \geq 1$, we see that $(qb_n - c_n)/a_n$ is increasing with $n \geq 2$, and by Lemma 4 it is seen that g_1 is increasing on $(0, \pi/2)$. Hence, we have

$$3q - \frac{8}{5} = \lim_{x \rightarrow 0^+} g_1(x) < g_1(x) < \lim_{x \rightarrow \pi/2^-} g_1(x) = \begin{cases} \infty & \text{if } q > 1, \\ \frac{\pi^2}{4} - 1 & \text{if } q = 1. \end{cases}$$

When $q \leq 34/35$, the sequence $(qb_n - c_n)/a_n$ is decreasing with $n \geq 2$, and so is the function $(qB - C)/A$ on $(0, \pi/2)$. Hence, we have (2.20).

This lemma is proved. \square

Lemma 7. Let g_2 be defined on $(0, \pi/2)$ by

$$(2.21) \quad g_2(x) = \frac{C(x) - \frac{8}{5}A(x)}{B(x) - 3A(x)}.$$

where $A(x)$, $B(x)$ and $C(x)$ are defined by (2.14a), (2.14b) and (2.14c), respectively. Then (i) $B(x) - 3A(x) > 0$ for $x \in (0, \pi/2)$; (ii) g_2 is increasing on $(0, \pi/2)$, and we have $34/35 < g_2(x) < 1$.

Proof. By using (2.15), (2.16) and (2.17), we get

$$g_2(x) = \frac{C(x) - \frac{8}{5}A(x)}{B(x) - 3A(x)} = \frac{\sum_{n=3}^{\infty} (c_n - \frac{8}{5}a_n)x^{2n}}{\sum_{n=3}^{\infty} (b_n - 3a_n)x^{2n}} \quad \text{and} \quad \frac{c_n - \frac{8}{5}a_n}{b_n - 3a_n} = \frac{2^{2n} - \frac{36}{5}n - \frac{8}{5}}{2^{2n} - 6n - 4}.$$

(i) In order for $B(x) - 3A(x) > 0$ to be true for $x \in (0, \pi/2)$, it suffices that $b_n - 3a_n > 0$ for $n \geq 3$. Employing binomial expansion yields

$$b_n - 3a_n = 2^{2n} - 6n - 4 > 1 + 2n + \frac{2n(2n-1)}{2} - 6n - 4 = (2n+1)(n-3) \geq 0.$$

(ii) By Lemma 4, to prove g_2 is increasing on $(0, \pi/2)$, it is enough to show that for $n \geq 3$

$$\frac{c_{n+1} - \frac{8}{5}a_{n+1}}{b_{n+1} - 3a_{n+1}} - \frac{c_n - \frac{8}{5}a_n}{b_n - 3a_n} > 0.$$

A direct computation leads to

$$\begin{aligned} \frac{c_{n+1} - \frac{8}{5}a_{n+1}}{b_{n+1} - 3a_{n+1}} - \frac{c_n - \frac{8}{5}a_n}{b_n - 3a_n} &= \frac{2^{2n+2} - \frac{36}{5}n - \frac{44}{5}}{2^{2n+2} - 6n - 10} - \frac{2^{2n} - \frac{36}{5}n - \frac{8}{5}}{2^{2n} - 6n - 4} \\ &= \frac{6}{5} \frac{(3n-7)2^{2n} + 16}{(2^{2n+2} - 6n - 10)(2^{2n} - 6n - 4)} > 0, \end{aligned}$$

which shows that the sequence $(c_n - 8a_n/5) / (b_n - 3a_n)$ is increasing with $n \geq 3$, and by Lemma 4 it is seen that g_2 is increasing on $(0, \pi/2)$. Consequently, we get

$$\frac{34}{35} = \lim_{x \rightarrow 0^+} g_2(x) < g_2(x) < \lim_{x \rightarrow \pi/2^-} g_2(x) = 1,$$

which proves the lemma. \square

Now we state and prove the monotonicity of $T_{p,q}$.

Proposition 1. *Let $T_{p,q}$ be defined on $(0, \pi/2)$ by (2.3). Then*

- (i) *when $q > 1$, $T_{p,q}$ is increasing on $(0, \pi/2)$ for $p \leq 3q - 8/5$;*
- (ii) *when $q = 1$, $T_{p,q}$ is increasing on $(0, \pi/2)$ for $p \leq 7/5$ and decreasing on $(0, \pi/2)$ for $p \geq \pi^2/4 - 1$;*
- (iii) *when $34/35 < q < 1$, $T_{p,q}$ is decreasing on $(0, \pi/2)$ for $p \geq \pi^2/4 - 1$;*
- (iv) *when $q \leq 34/35$, $T_{p,q}$ is decreasing on $(0, \pi/2)$ for $p \geq 3q - 8/5$.*

Proof. As mentioned previously, to derive the monotonicity of $T_{p,q}$, it suffices to deal with the sings of $f_2(x)$ on $(0, \pi/2)$. To this end, we need to write $f_2(x)$ in the form of

$$(2.22) \quad f_2(x) = pA(x) - qB(x) + C(x) = A(x)(p - g_1(x)),$$

where $g_1(x)$ is defined by (2.18). Then, $\text{sgn } f_2(x) = \text{sgn}(p - g_1(x))$ due to $A(x) > 0$ for $x \in (0, \pi/2)$.

(i) When $q > 1$, it is obtained from Lemma 6 that $f_2(x) < 0$ for $x \in (0, \pi/2)$ due to

$$p - g_1(x) \leq p - \left(3q - \frac{8}{5}\right) \leq 0$$

Utilizing the relation (2.12) and Lemma 2 we get the first assertion in this theorem.

(ii) When $q = 1$, similarly, it is acquired that $f_2(x) < 0$ due to $(p - g_1(x)) \leq 3 \times 1 - 8/5 = 7/5$ and $f_2(x) > 0$ due to

$$p - g_1(x) \geq p - \left(\frac{\pi^2}{4} - 1\right) \geq 0.$$

Make use of the relation (2.12) and Lemma 2 again, the second assertion in this theorem follows.

(iii) When $34/35 < q < 1$, we see that $f_2(x) > 0$ in view of

$$p - g_1(x) = p - \frac{qB(x) - C(x)}{A(x)} > p - \frac{1 \times B(x) - C(x)}{A(x)} \geq p - \left(\frac{\pi^2}{4} - 1\right) \geq 0.$$

(iv) When $q \leq 34/35$, it can be proved in the same way.

Thus we complete the proof. \square

Letting $p = 3q - 8/5$ in Proposition 1, we have

Corollary 1. *Let $T_{p,q}$ be defined on $(0, \pi/2)$ by (2.3). Then $T_{3q-8/5,q}$ is increasing on $(0, \pi/2)$ for $q \geq 1$ and decreasing for $q \leq 34/35$.*

Since $p \leq (\geq) 3q - 8/5$ is equivalent to $q \geq (\leq) p/3 + 8/15$, Proposition 1 can be restated as follows.

Proposition 2. *Let $T_{p,q}$ be defined on $(0, \pi/2)$ by (2.3). Then*

- (i) $T_{p,q}$ is increasing on $(0, \pi/2)$ if $q \geq \max(1, p/3 + 8/15)$;
- (ii) $T_{p,q}$ is decreasing on $(0, \pi/2)$ if $q \leq \min(34/35, p/3 + 8/15)$ or $p \geq \pi^2/4 - 1$ and $34/35 < q \leq 1$.

Due to

$$\begin{aligned} \max\left(1, \frac{p}{3} + \frac{8}{15}\right) &= \begin{cases} \frac{p}{3} + \frac{8}{15} & \text{if } p \geq \frac{7}{5}, \\ 1 & \text{if } p < \frac{7}{5}, \end{cases} \\ \min\left(\frac{34}{35}, \frac{p}{3} + \frac{8}{15}\right) &= \begin{cases} \frac{34}{35} & \text{if } p \geq \frac{46}{35}, \\ \frac{p}{3} + \frac{8}{15} & \text{if } p < \frac{46}{35}, \end{cases} \end{aligned}$$

Position 2 also can be restated in another equivalent assertion.

Proposition 3. *Let $T_{p,q}$ be defined on $(0, \pi/2)$ by (2.3). Then*

- (i) if $p \geq \pi^2/4 - 1$, then $T_{p,q}$ is increasing on $(0, \pi/2)$ for $q \geq p/3 + 8/15$ and decreasing on $(0, \pi/2)$ for $q \leq 1$;
- (ii) if $p \in [7/5, \pi^2/4 - 1)$, then $T_{p,q}$ is increasing on $(0, \pi/2)$ for $q \geq p/3 + 8/15$ and decreasing on $(0, \pi/2)$ for $q \leq 34/35$;
- (iii) if $p \in [46/35, 7/5)$, then $T_{p,q}$ is increasing on $(0, \pi/2)$ for $q \geq 1$ and decreasing on $(0, \pi/2)$ for $q \leq 34/35$;
- (iv) if $p < 46/35$, then $T_{p,q}$ is increasing on $(0, \pi/2)$ for $q \geq 1$ and decreasing on $(0, \pi/2)$ for $q \leq p/3 + 8/15$.

Let $p = kq$. Then solving the simultaneous inequalities $q \geq \max(1, kq/3 + 8/15)$ and $q \leq \min(34/35, kq/3 + 8/15)$ for q give

$$\begin{cases} q \geq \frac{8}{5(3-k)} & \text{if } k \in [\frac{7}{5}, 3), \\ q \geq 1 & \text{if } k \in (-\infty, \frac{7}{5}] \end{cases} \quad \text{and} \quad \begin{cases} \frac{8}{5(3-k)} \leq q \leq 34/35 & \text{if } k \in (3, \infty), \\ q \leq \frac{34}{35} & \text{if } k \in [\frac{23}{17}, 3), \\ q \leq \frac{8}{5(3-k)} & \text{if } k \in (-\infty, \frac{23}{17}], \end{cases}$$

respectively; while the solution of the simultaneous inequalities $kq \geq \pi^2/4 - 1$ and $34/35 < q \leq 1$ is:

$$\begin{cases} 34/35 < q \leq 1 & \text{if } k \geq \frac{35\pi^2 - 140}{136} \approx 1.5106, \\ \frac{\pi^2 - 4}{4k} \leq q \leq 1 & \text{if } k \in \left(\frac{\pi^2 - 4}{4}, \frac{35\pi^2 - 140}{136}\right). \end{cases}$$

By Proposition 2, we have

Corollary 2. *Let $T_{p,q}$ be defined on $(0, \pi/2)$ by (2.3). Then*

- (i) when $k \in (3, \infty)$, $T_{kq,q}$ is decreasing for $8/(5(3-k)) \leq q \leq 34/35$;
- (ii) when $k \in [(35\pi^2 - 140)/136, 3)$, $T_{kq,q}$ is increasing for $q \geq 8/(5(3-k))$ and decreasing for $q \leq 1$;
- (iii) when $k \in [\pi^2/4 - 1, (35\pi^2 - 140)/136)$, $T_{kq,q}$ is increasing for $q \geq 8/(5(3-k))$ and decreasing for $q \leq 34/35$ or $(\pi^2/4 - 1)/k \leq q \leq 1$;
- (iv) when $k \in [7/5, \pi^2/4 - 1)$, $T_{kq,q}$ is increasing for $q \geq 8/(5(3-k))$ and decreasing for $q \leq 34/35$;
- (v) when $k \in [23/17, 7/5)$, $T_{kq,q}$ is increasing for $q \geq 1$ and decreasing for $q \leq 34/35$;
- (vi) when $k \in (-\infty, 23/17)$, $T_{kq,q}$ is increasing for $q \geq 1$ and decreasing for $q \leq 8/(5(3-k))$.

3. RESULTS AND PROOFS

In this section, we will give some new inequalities involving trigonometric functions by using monotonicity theorems given in previous section. For clarity of expressions, we will directly write $S_p(x)$, $C_q(x)$, $T_{p,q}(x)$ etc. by their general formulas, and if $pq = 0$, then we regard them as limits at $p = 0$ or $q = 0$, unless otherwise specified.

3.1. In the general case. A simple computation yields

$$T_{p,q}(0^+) = \frac{1}{3} \quad \text{and} \quad T_{p,q}\left(\frac{\pi^-}{2}\right) = \begin{cases} \frac{q}{p} \left(1 - \left(\frac{2}{\pi}\right)^p\right) & \text{if } q > 0, p \neq 0, \\ -q \ln \frac{2}{\pi} & \text{if } q > 0, p = 0, \\ 0 & \text{if } q \leq 0. \end{cases}$$

And then, by Proposition 1, we obtain the following theorem.

Theorem 1. *Let $x \in (0, \pi/2)$.*

(i) *If $q \geq 1$ and $p \leq 3q - 8/5$, then the inequalities*

$$(3.1) \quad \left(\frac{2}{\pi}\right)^p + \left(1 - \left(\frac{2}{\pi}\right)^p\right) \cos^q x < \left(\frac{\sin x}{x}\right)^p < 1 - \frac{p}{3q} + \frac{p}{3q} \cos^q x \quad \text{if } p > 0,$$

$$(3.2) \quad \left(\frac{2}{\pi}\right)^{1-\cos^q x} < \frac{\sin x}{x} < \exp \frac{\cos^q x - 1}{3q} \quad \text{if } p = 0,$$

$$(3.3) \quad 1 - \frac{p}{3q} + \frac{p}{3q} \cos^q x < \left(\frac{\sin x}{x}\right)^p < \left(\frac{2}{\pi}\right)^p + \left(1 - \left(\frac{2}{\pi}\right)^p\right) \cos^q x \quad \text{if } p < 0$$

hold, where $1/3$ and $q(1 - (2/\pi)^p)/p$ are the best constants.

(ii) *If $34/35 < q \leq 1$ and $p \geq \pi^2/4 - 1$, then the double inequalities (3.1) is reversed.*

(iii) *If $0 < q \leq 34/35$ and $p \geq 3q - 8/5$, then all the double inequalities (3.1), (3.2) and (3.3) are reversed.*

(iv) *If $q \leq 0$ and $p \geq 3q - 8/5$, then the inequalities*

$$\begin{aligned} \left(\frac{\sin x}{x}\right)^p &> 1 - \frac{p}{3q} + \frac{p}{3q} \cos^q x \quad \text{if } p > 0, \\ \frac{\sin x}{x} &> \exp \frac{\cos^q x - 1}{3q} \quad \text{if } p = 0, \\ \left(\frac{\sin x}{x}\right)^p &< 1 - \frac{p}{3q} + \frac{p}{3q} \cos^q x \quad \text{if } p < 0, \end{aligned}$$

hold, where $1/3$ is the best constant.

Proof. We only prove (i), others can be proved in the same way. By part (i) of Proposition 1, if $q \geq 1$ and $p \leq 3q - 8/5$, then we get $T_{p,q}(0^+) < T_{p,q}(x) < T_{p,q}(\pi/2^-)$, that is,

$$\frac{1}{3} C_q(x) < S_p(x) < T_{p,q}(\pi/2^-) C_q(x),$$

which is equivalent to (3.1), (3.2) and (3.3). This completes the proof. \square

Remark 1. *Letting $p = q$ in Theorem 1 yields Theorem Zhu 1. It can be seen that our result is a generalization of Zhu's [7].*

Remark 2. *For $0 < a < b$ and $(w, q) \in \Omega_{w,q} = \{q \geq 0, w \leq 1 \text{ or } q \leq 0, w \geq 0\}$, we define M_q by*

$$(3.4) \quad M_q(a, b; w) = (wa^q + (1-w)b^q)^{1/q} \quad \text{if } q \neq 0 \quad \text{and} \quad M_0(a, b; w) = a^w b^{1-w}.$$

It is clear that $M_q(a, b; w)$ is a weighted power mean of order q of a and b for $w \in (0, 1)$, but not a mean of a and b for $w \geq 1, q \leq 0$ or $w \leq 0, q \geq 0$ because that

$$M_q(a, b; w) \leq a \text{ for } w \geq 1, q \leq 0 \text{ and } M_q(a, b; w) \geq b \text{ for } w \leq 0, q \geq 0.$$

When $w = p/(3q), 1 - (2/\pi)^p$, that $(w, q) \in \Omega_{w,q}$ implies that

$$(p, q) \in E_{p,q} = \{p \leq 0 \text{ or } 0 < p \leq 3q\} \text{ and } (p, q) \in \mathbb{R} \times \mathbb{R}_+,$$

respectively. Thus, inequalities (3.1), (3.2) and (3.3) can be unified into one:

$$(3.5) \quad M_q^{q/p}(\cos x, 1; 1 - (2/\pi)^p) < \frac{\sin x}{x} < M_q^{q/p}(\cos x, 1; p/(3q)).$$

For convenience, we denote by

$$M_q^{q/p}(\cos x, 1; p/(3q)) = M(\cos x; p, q) \text{ and } M_q^{q/p}(\cos x, 1; 1 - (2/\pi)^p) = N(\cos x; p, q).$$

Then,

$$(3.6) \quad M(t; p, q) = \begin{cases} \left(1 - \frac{p}{3q} + \frac{p}{3q}t^q\right)^{1/p} & \text{if } pq \neq 0, (p, q) \in E_{p,q}, \\ \exp \frac{t^q - 1}{3q} & \text{if } p = 0, q \neq 0, \\ \left(\frac{p}{3} \ln t + 1\right)^{1/p} & \text{if } p < 0, q = 0, \\ t^{1/3} & \text{if } p = q = 0, \end{cases}$$

$$(3.7) \quad N(t; p, q) = \begin{cases} \left(\left(\frac{2}{\pi}\right)^p + \left(1 - \left(\frac{2}{\pi}\right)^p\right)t^q\right)^{1/p} & \text{if } p \neq 0, \\ \left(\frac{2}{\pi}\right)^{1-t^q} & \text{if } p = 0. \end{cases}$$

And then, we have the following assertions:

(i) For $x \in (0, \pi/2)$, if $(p, q) \in E_{p,q} = \{(p, q) : p \leq 0 \text{ or } 0 < p \leq 3q\}$, then

$$T_{p,q}(x) > (<) T_{p,q}(0^+) \iff \frac{\sin x}{x} < (>) M(\cos x; p, q).$$

(ii) For $x \in (0, \pi/2)$, if $(p, q) \in \mathbb{R} \times \mathbb{R}_+$, then

$$T_{p,q}(x) < (>) T_{p,q}(\pi/2^-) \iff \frac{\sin x}{x} > (<) N(\cos x; p, q).$$

Remark 3. (i) For the monotonicity of $M(t; p, q)$ with respect to p, q , we suggest that:

Let $E_{p,q} = \{(p, q) : p \leq 0 \text{ or } 0 < p \leq 3q\}$ and let M be the function defined on $(0, 1) \times E_{p,q}$ by (3.6). Then M is decreasing in p and increasing in q .

Indeed, for $pq \neq 0$, logarithmic differentiation yields

$$\begin{aligned} \frac{\partial \ln M}{\partial p} &= \frac{1}{p^2} \left(-\ln \left(\frac{p}{3q}t^q + 1 - \frac{p}{3q} \right) - \frac{p(1-t^q)}{(pt^q + 3q - p)} \right) := \frac{1}{p^2} M_1(t; p, q), \\ \frac{\partial M_1}{\partial p} &= -\frac{p(1-t^q)^2}{(pt^q + 3q - p)^2}, \end{aligned}$$

which implies that M_1 is decreasing in p on $(0, \infty)$ and increasing on $(-\infty, 0)$. Hence we have $M_1(t; p, q) < M_1(t; 0, q) = 0$, and so M is decreasing in p .

In the case of $pq \neq 0, (p, q) \in E_{p,q}$, it can be proved in the same way.

Similarly, we have

$$\frac{\partial \ln M}{\partial q} = -\frac{t^q}{3q^2 \left(\frac{p}{3q}t^q + 1 - \frac{p}{3q} \right)} (\ln t^{-q} - (t^{-q} - 1)) > 0,$$

where the last inequality holds due to $\ln x \leq x-1$ for $x > 0$ and $((p/(3q))t^q + 1 - (p/(3q))) > 0$ for $(t, p, q) \in (0, 1) \times E_{p,q}$, which proves the monotonicity of M with respect to q .

(ii) For the monotonicity of $N(t; p, q)$ with respect to p, q , we claim that:

Let N be defined on $(0, 1) \times \mathbb{R} \times \mathbb{R}_+$ by (3.7). Then N is increasing in p and decreasing in q .

In fact, $N(t; p, q)$ can be written as

$$N(t; p, q) = \left(t^q + (1 - t^q) \left(\frac{2}{\pi} \right)^p \right)^{1/p} \quad \text{if } p \neq 0 \quad \text{and} \quad N(t; 0, q) = \left(\frac{2}{\pi} \right)^{1-t^q},$$

which is clearly a weighted power mean of order p of positive numbers 1 and $2/\pi$, and consequently, N is increasing with respect to p .

The decreasing property of N in q can be derived from that for $p \neq 0$,

$$\frac{\partial N}{\partial q} = \left(t^q + (1 - t^q) \left(\frac{2}{\pi} \right)^p \right)^{1/p-1} \times \frac{1 - \left(\frac{2}{\pi} \right)^p}{p} \times t^q \ln t < 0,$$

where the inequality is valid because that $(1 - (2/\pi)^p)/p > 0$ and $t \in (0, 1)$.

Utilizing Proposition 3, the following theorem is immediate.

Theorem 2. Let $x \in (0, \pi/2)$. Then

(i) if $p \geq \pi^2/4 - 1$, then the double inequality

$$(3.8) \quad \frac{1 - (\cos x)^{q_2}}{3q_2} < \frac{1 - \left(\frac{\sin x}{x} \right)^p}{p} < \frac{1 - (\cos x)^{q_1}}{3q_1}$$

holds for $q_2 \geq p/3 + 8/15$ and $q_1 \leq 1$;

(ii) if $p \in [7/5, \pi^2/4 - 1)$, then (3.8) holds for $q_2 \geq p/3 + 8/15$ and $q_1 \leq 34/35$;

(iii) if $p \in [46/35, 7/5)$, then (3.8) holds for $q_2 \geq 1$ and $q_1 \leq 34/35$;

(iv) if $p < 46/35$, then (3.8) holds for $q_2 \geq 1$ and for $q_1 \leq p/3 + 8/15$.

Remark 4. If $(p, q) \in E_{p,q} = \{(p, q) : p \leq 0 \text{ or } 0 < p \leq 3q\}$, then Theorem 2 can be restated as follows: Let $x \in (0, \pi/2)$. Then

(i) if $p \geq \pi^2/4 - 1$, then the double inequality

$$(3.9) \quad \left(1 - \frac{p}{3q_1} + \frac{p}{3q_1} \cos^{q_1} x \right)^{1/p} < \frac{\sin x}{x} < \left(1 - \frac{p}{3q_2} + \frac{p}{3q_2} \cos^{q_2} x \right)^{1/p}$$

holds for $p/3 \leq q_1 \leq 1$ and $q_2 \geq p/3 + 8/15$;

(ii) if $p \in [7/5, \pi^2/4 - 1)$, then (3.9) holds for $p/3 \leq q_1 \leq 34/35$ and $q_2 \geq p/3 + 8/15$;

(iii) if $p \in (46/35, 7/5)$, then (3.9) holds for $p/3 \leq q_1 \leq 34/35$ and $q_2 \geq 1$;

(iv) if $p \in (0, 46/35]$, then (3.9) holds for $p/3 \leq q_1 \leq p/3 + 8/15$ and $q_2 \geq 1$;

(v) if $p \leq 0$, then (3.9) holds for $q_1 \leq p/3 + 8/15$ and $q_2 \geq 1$.

Before showing the sharp inequalities for trigonometric functions, we give a useful lemma.

Lemma 8. Let $q > 0$ and let $D_{p,q}$ be defined on $(0, \pi/2)$ by

$$(3.10) \quad D_{p,q}(x) = T_{p,q}(x) - \frac{1}{3} = \frac{S_p(x)}{C_q(x)} - \frac{1}{3}.$$

(i) We have

$$(3.11) \quad \lim_{x \rightarrow 0^+} \frac{D_{p,q}(x)}{x^2} = -\frac{1}{36} \left(p - 3q + \frac{8}{5} \right),$$

$$(3.12) \quad \lim_{x \rightarrow 0^+} \frac{D_{3q-8/5,q}(x)}{x^4} = \frac{1}{135} \left(q - \frac{34}{35} \right),$$

$$(3.13) \quad D_{p,q}(\pi/2^-) = \begin{cases} \frac{q}{p} \left(1 - \left(\frac{2}{\pi} \right)^p \right) - \frac{1}{3} & \text{if } q > 0, p \neq 0, \\ -q \ln \frac{2}{\pi} - \frac{1}{3} & \text{if } q > 0, p = 0, \\ -\frac{1}{3} & \text{if } q \leq 0. \end{cases}$$

(ii) If $q > 0$, then for fixed $p > 0$, the equation $D_{p,q}(\pi/2^-) = 0$ has a unique root $q(p)$ on \mathbb{R} such that $D_{p,q}(\pi/2^-) > 0$ for $q > q(p)$ and $D_{p,q}(\pi/2^-) < 0$ for $q < q(p)$, where

$$(3.14) \quad q(p) = \frac{p}{3(1 - (2/\pi)^p)} \text{ if } p \neq 0 \text{ and } q(0) = -\frac{1}{3 \ln(2/\pi)}.$$

(iii) For fixed $q > 0$, the equation $D_{p,q}(\pi/2^-) = 0$ has a unique root $p(q)$ on \mathbb{R} such that $D_{p,q}(\pi/2^-) > 0$ for $p < p(q)$ and $D_{p,q}(\pi/2^-) < 0$ for $p > p(q)$, where $p(q)$ is the inverse function of $q(p)$. In particular, we have

$$(3.15) \quad p_0 = p(1) \approx 1.42034 \text{ and } p_0^* = p\left(\frac{34}{35}\right) \approx 1.27754;$$

(iv) Both functions $q \mapsto p(q)$ and $p \mapsto q(p)$ are increasing.

Proof. (i) For $pq \neq 0$, expanding in power series yields

$$D_{p,q}(x) = -\frac{5p-15q+8}{180}x^2 + \frac{70p^2+315q^2-315pq+126p+126q-304}{45360}x^4 + o(x^6),$$

which leads to (3.11). It is easy to check that it holds for $p = 0$ or $q = 0$.

If $p = 3q - 8/5$, then we have

$$D_{p,q}(x) = \frac{35q-34}{4725}x^4 + o(x^6),$$

which implies (3.12).

(ii) If $q > 0$, then for fixed $p > 0$, solving the equation $D_{p,q}(\pi/2^-) = 0$ for q we get $q = q(p)$, where $q(p)$ is defined by (3.14). It is easy to check that $D_{p,q}(\pi/2^-) > 0$ for $q > q(p)$ and $D_{p,q}(\pi/2^-) < 0$ for $q < q(p)$.

(iii) For fixed $q > 0$, by Lemma 1, we see that $p \mapsto D_{p,q}(x)$ is decreasing on \mathbb{R} , which together with the facts that

$$D_{-\infty,q}\left(\frac{\pi^-}{2}\right) = \infty \text{ and } D_{\infty,q}\left(\frac{\pi^-}{2}\right) = -\frac{1}{3} < 0$$

gives the desired assertion. Clearly, as a unique root of the equation $D_{p,q}(\pi/2^-) = 0$, $p = p(q)$ is the inverse function of $q(p)$. By mathematical computer software we can find the approximations of $p(1)$ and $p(34/35)$.

(iv) From Lemma 1 it is easy to see that $q \mapsto p(q)$ is increasing, and so is its inverse.

This lemma is proved. \square

Now we are ready to present sharp bounds for $(\sin x)/x$ in terms of $M(\cos x; p, q)$ when p is fixed.

Theorem 3. Let $q(p)$ be defined by (3.14) and $(p, q) \in E_{p,q} = \{(p, q) : p \leq 0 \text{ or } 0 < p \leq 3q\}$.

(i) If $p \geq 7/5$, then the inequality

$$(3.16) \quad \frac{\sin x}{x} < \left(1 - \frac{p}{3q} + \frac{p}{3q} \cos^q x\right)^{1/p}$$

holds for $x \in (0, \pi/2)$ if and only if $q \geq p/3 + 8/15$.

(ii) If $p \geq p_0 \approx 1.42034$, where p_0 is defined by (3.15), then the inequality

$$(3.17) \quad \frac{\sin x}{x} > \left(1 - \frac{p}{3q} + \frac{p}{3q} \cos^q x\right)^{1/p}$$

holds for $x \in (0, \pi/2)$ if and only if $q \leq q(p)$;

(iii) if $p \leq 46/35$, then the inequality (3.17) holds for $x \in (0, \pi/2)$ if and only if $q \leq p/3 + 8/15$.

(iv) If $p \leq p_0^* \approx 1.27754$, where p_0^* is defined by (3.15), then (3.16) holds for $x \in (0, \pi/2)$ if and only if $q \geq q(p)$.

Proof. As shown in Remark 2, if $(p, q) \in E_{p,q}$, then inequality (3.16) or (3.17) holds if and only if $D_{p,q}(x) := S_p(x)/C_q(x) - 1/3 > 0$ (or < 0).

(i) When $p \geq 7/5$, we prove the inequality (3.16) holds for $x \in (0, \pi/2)$ if and only if $q \geq p/3 + 8/15$. The necessity is obtained from

$$\lim_{x \rightarrow 0^+} x^{-2} D_{p,q}(x) = \frac{1}{24} \left(q - \left(\frac{p}{3} + \frac{8}{15} \right) \right) \geq 0,$$

which gives $q \geq p/3 + 8/15$. The sufficiency easily follows by parts (i) and (ii) of Remark 4.

(ii) When $p \geq p_0 \approx 1.42034$ in which p_0 satisfies that $p_0(1 - (2/\pi)^{p_0})^{-1}/3 = 1$, we show that the inequality (3.17) holds for $x \in (0, \pi/2)$ if and only if $p/3 \leq q \leq p(1 - (2/\pi)^p)^{-1}/3$. The necessity can be derived from $\lim_{x \rightarrow \pi/2^-} D_{p,q}(x) \leq 0$, which by Lemma 8 yields $q \leq p(1 - (2/\pi)^p)^{-1}/3 = q(p)$.

Now we prove the sufficiency. Due to Remark 3, it is seen that $q \mapsto M(\cos x; p, q)$ is increasing, and it suffices to show that the inequality (3.17) holds for $x \in (0, \pi/2)$ when $q = q(p)$. Also, by part (iii) of Lemma 8, $p \mapsto q(p) = p(1 - (2/\pi)^p)^{-1}/3$ is increasing, so we get $q = q(p) \geq q(p_0) = p_0(1 - (2/\pi)^{p_0})^{-1}/3 = 1$. From Lemma 6, when $q \geq 1$, the function $g_1 = (q(p)B - C)/A$ is increasing on $(0, \pi/2)$, and so $x \mapsto p - g_1(x) := h(x, p, q(p))$ is decreasing on $(0, \pi/2)$. But,

$$\begin{aligned} h(0^+, p, q(p)) &= p - \left(3q(p) - \frac{8}{5} \right) = p - \frac{p}{1 - (2/\pi)^p} + \frac{8}{5}, \\ h\left(\frac{\pi^-}{2}, p, q(p)\right) &= \begin{cases} p - \infty & \text{if } q(p) > 1, \\ p - \left(\frac{\pi^2}{4} - 1\right) < 0 & \text{if } q(p) = 1, \end{cases} \end{aligned}$$

where $p - (\pi^2/4 - 1) < 0$ due to that $q(p) = 1$ implies $p = p_0 \approx 1.42034$. Also, we claim that $h(0^+, p, q(p)) > 0$ for $p \geq p_0$. In fact, differentiation leads to

$$h'(0^+, p, q(p)) = -\frac{(2/\pi)^p}{(1 - (2/\pi)^p)^2} \left(\ln \left(\frac{2}{\pi} \right)^p - \left(\frac{2}{\pi} \right)^p + 1 \right) \geq 0,$$

where the last inequality holds due to $\ln x \leq x - 1$ for $x > 0$. Hence,

$$h(0^+, p, q(p)) \geq h(0^+, p_0, q(p_0)) = p_0 - \frac{p_0}{1 - (2/\pi)^{p_0}} + \frac{8}{5} = p_0 - 3 + \frac{8}{5} > 0.$$

Therefore, there is a unique number $x_0 \in (0, \pi/2)$ such that $h(x, p, q(p)) > 0$ for $x \in (0, x_0)$ and $h(x, p, q(p)) < 0$ for $x \in (x_0, \pi/2)$, which together with (2.22) and (2.12) means that the function $x \mapsto S'_p(x)/C'_q(x)$ is decreasing on $(0, x_0]$ and increasing on $(x_0, \pi/2)$. We note that $C'_q(x) = \cos^{q-1} x \sin x > 0$ for $x \in (0, \pi/2)$ and the relation

$$\frac{S_p(\frac{\pi}{2}^-) - S_p(0^+)}{C_q(\frac{\pi}{2}^-) - C_q(0^+)} = \frac{1}{3}$$

holds, utilizing Lemma 3 it is derived that the inequality

$$\frac{S_p(x) - S_p(0^+)}{C_q(x) - C_q(0^+)} < \frac{1}{3}$$

holds for all $x \in (0, \pi/2)$, that is, $S_p(x)/C_q(x) < 1/3$ is valid for $x \in (0, \pi/2)$, which prove the sufficiency.

(iii) When $p \leq 46/35$, we prove the inequality of (3.17) holds $x \in (0, \pi/2)$ if and only if $q \leq p/3 + 8/15$ and $(p, q) \in E_{p,q}$. The necessity easily follows from

$$\lim_{x \rightarrow 0^+} x^{-2} D_{p,q}(x) = \frac{1}{24} \left(q - \left(\frac{p}{3} + \frac{8}{15} \right) \right) \leq 0,$$

which gives $q \leq p/3 + 8/15$. The sufficiency easily follows by part (iv) and (v) of Remark 4.

(iv) Finally, we prove that when $p \leq p_0^* \approx 1.27754$ in which p_0^* satisfies that $p_0^* \left(1 - (2/\pi)^{p_0^*} \right)^{-1} / 3 = 34/35$, the inequality (3.16) holds for $x \in (0, \pi/2)$ if and only if $q \geq q(p)$. The necessity can be derived from $\lim_{x \rightarrow \pi/2^-} D_{p,q}(x) \geq 0$, which by Lemma 8 leads us to $q \geq q(p)$.

Similarly, to prove the sufficiency, it suffices to show that the inequality (3.16) holds for $x \in (0, \pi/2)$ when $q = q(p)$. Also, by part (iii) of Lemma 8, $p \mapsto q(p)$ is increasing, so we get $q(p) \leq q(p_0^*) = p_0^* \left(1 - (2/\pi)^{p_0^*} \right)^{-1} / 3 = 34/35$. By Lemma 6, when $q \leq 34/35$, the function $g_1 = (qB - C)/A$ is decreasing on $(0, \pi/2)$, and so $x \mapsto p - g_1(x) := h(x, p, q(p))$ is increasing on $(0, \pi/2)$. But,

$$\begin{aligned} h(0^+, p, q(p)) &= p - \left(3q(p) - \frac{8}{5} \right) = p - \frac{p}{1 - (2/\pi)^p} + \frac{8}{5}, \\ h\left(\frac{\pi}{2}^-, p, q(p)\right) &= p + \infty. \end{aligned}$$

As shown previously, $p \mapsto h(0^+, p, q(p))$ is increasing, and so for $p \leq p_0^*$

$$h(0^+, p, q(p)) \leq h(0^+, p_0^*, q(p_0^*)) = p_0^* - \left(3 \times \frac{34}{35} - \frac{8}{5} \right) = p_0^* - \frac{46}{35} < 0,$$

Thus, there is a unique number $x_1 \in (0, \pi/2)$ such that $h(x, p, q(p)) < 0$ for $x \in (0, x_1)$ and $h(x, p, q(p)) > 0$ for $x \in (x_1, \pi/2)$, which together with (2.22) and (2.12) means that the function $x \mapsto S'_p(x)/C'_q(x)$ is increasing on $(0, x_1]$ and decreasing on $(x_1, \pi/2)$. Similar to part (ii) of this proof, utilizing Lemma 3 again we see that the inequality $S_p(x)/C_q(x) > 1/3$ holds true for $x \in (0, \pi/2)$, which prove the sufficiency.

This completes the proof of this theorem. \square

Letting $p = 7/5$ and $p = p_0 \approx 1.42034$ in Theorem 3 we have

Corollary 3. *For $x \in (0, \pi/2)$, the double inequality*

$$\left(1 - \frac{p_0}{3q_1} + \frac{p_0}{3q_1} \cos^{q_1} x\right)^{1/p_0} < \frac{\sin x}{x} < \left(1 - \frac{7}{15q_2} + \frac{7}{15q_2} \cos^{q_2} x\right)^{5/7}$$

holds if and only if $p_0/3 \leq q_1 \leq 1$ and $q_2 \geq 1$, where $p_0 \approx 1.42034$ satisfies that $p_0(1 - (2/\pi)^{p_0})^{-1}/3 = 1$. Particularly, taking $q_1 = q_2 = 1$ we have

$$\left(\frac{3-p_0}{3} + \frac{p_0}{3} \cos x\right)^{1/p_0} < \frac{\sin x}{x} < \left(\frac{8}{15} + \frac{7}{15} \cos x\right)^{5/7},$$

where $p_0 \approx 1.42034$ and $7/5 = 1.4$ are the best.

Letting $p = 46/35$ and $p = p_0^* \approx 1.27754$ in Theorem 3 we get

Corollary 4. *For $x \in (0, \pi/2)$, the double inequality*

$$\left(1 - \frac{46}{105q_1} + \frac{46}{105q_1} \cos^{q_1} x\right)^{35/46} < \frac{\sin x}{x} < \left(1 - \frac{p_0^*}{3q_2} + \frac{p_0^*}{3q_2} \cos^{q_2} x\right)^{1/p_0^*}$$

holds if and only if $46/105 \leq q_1 \leq 34/35$ and $q_2 \geq 34/35$, where $p_0^ \approx 1.27754$ satisfies that $p_0^*(1 - (2/\pi)^{p_0^*})^{-1}/3 = 34/35$. Particularly, putting $q_1 = q_2 = 34/35$ we have*

$$\left(\frac{28}{51} + \frac{23}{51} \cos^{34/35} x\right)^{35/46} < \frac{\sin x}{x} < \left(\frac{102 - 35p_0^*}{102} + \frac{35p_0^*}{102} \cos^{34/35} x\right)^{1/p_0^*},$$

where $46/35 \approx 1.3143$ and $p_0^ \approx 1.27754$ are the best.*

Letting $p = 0, 1$ in Theorem 3 we get

Corollary 5. *(i) The double inequality*

$$\exp\left(\frac{\cos^{q_1} x - 1}{3q_1}\right) < \frac{\sin x}{x} < \exp\left(\frac{\cos^{q_2} x - 1}{3q_2}\right)$$

holds for $x \in (0, \pi/2)$ if and only if $q_1 \leq 8/15$ and $q_2 \geq (3 \ln(\pi/2))^{-1} \approx 0.73814$.

(ii) The double inequality

$$(3.18) \quad 1 - \frac{1}{3q_1} + \frac{1}{3q_1} \cos^{q_1} x < \frac{\sin x}{x} < 1 - \frac{1}{3q_2} + \frac{1}{3q_2} \cos^{q_2} x$$

holds for $x \in (0, \pi/2)$ if and only if $1/3 \leq q_1 \leq 13/15$ and $q_2 \geq \pi/(3\pi - 6) \approx 0.91731$.

Remark 5. *Letting $q_1 = 13/15, 2/3, 1/2, 1/3$ and $q_2 = 1$ and using the increasing property of $M(\cos x; p, q)$ in q , we get the following chain of inequalities from (3.18):*

$$\begin{aligned} \cos^{1/3} x &< \frac{1}{3} + \frac{2}{3} \cos^{1/2} x < \frac{1}{2} + \frac{1}{2} \cos^{2/3} x \\ &< \frac{8}{13} + \frac{5}{13} \cos^{13/15} x < \frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3} \cos x. \end{aligned}$$

Theorem 4. Let $p(q)$ be the unique root of equation $D_{p,q}(\pi/2^-) = 0$ for fixed $q > 0$ and $(p_i, q) \in E_{p_i,q} = \{(p_i, q) : p_i \leq 0 \text{ or } 0 < p_i \leq 3q\}$, $i = 1, 2$.

(i) If $q \geq 1$, then the double inequality

$$(3.19) \quad \left(1 - \frac{p_1}{3q} + \frac{p_1}{3q} \cos^q x\right)^{1/p_1} < \frac{\sin x}{x} < \left(1 - \frac{p_2}{3q} + \frac{p_2}{3q} \cos^q x\right)^{1/p_2}$$

holds for $x \in (0, \pi/2)$ if and only if $p_1 \geq p(q)$ and $p_2 \leq 3q - 8/5$.

(ii) If $0 < q \leq 34/35$, then the double inequality (3.19) holds if and only if $p_1 \geq 3q - 8/5$ and $p_2 \leq p(q)$.

(iii) If $q \leq 0$, then the first inequality in (3.19) holds if and only if $p_1 \geq 3q - 8/5$.

Proof. In the case of $q \geq 1$. For $(p_2, q) \in E_{p_2,q}$, the second inequality in (3.19) is equivalent to $D_{p_2,q}(x) := S_{p_2}(x)/C_q(x) - 1/3 > 0$ for $x \in (0, \pi/2)$. If it holds for all $x \in (0, \pi/2)$, then we have

$$\lim_{x \rightarrow 0^+} x^{-2} D_{p_2,q}(x) = \frac{1}{24} \left(q - \left(\frac{p_2}{3} + \frac{8}{15} \right) \right) \geq 0,$$

which yields $p_2 \leq 3q - 8/5$. Conversely, if $p_2 \leq 3q - 8/5$, then by part (i) of Proposition 1 we get that $T_{p_2,q} = S_{p_2}(x)/C_q(x)$ is increasing on $(0, \pi/2)$, and so $S_{p_2}(x)/C_q(x) > S_{p_2}(0^+)/C_q(0^+) = 1/3$, which implies the second inequality in (3.19).

For $(p_1, q) \in E_{p_1,q}$, if the first inequality in (3.19) holds for all $x \in (0, \pi/2)$, that is, $D_{p_1,q}(x) := S_{p_1}(x)/C_q(x) - 1/3 < 0$, then there must be $D_{p_1,q}(\pi/2^-) \leq 0$, and from Lemma 8 we get $p_1 \geq p(q)$, where $p(q)$ is the inverse function of $q(p)$ defined by (3.14).

Now we prove the condition $p_1 \geq p(q)$ is sufficient for the first inequality in (3.19) to hold. Lemma 8 tell us that $q \mapsto p(q)$ is increasing, which together with $q \geq 1$ gives $p_1 \geq p(q) \geq p(1) = p_0$. And, $p_1 > p(q)$ means $q < q(p_1)$. From part (ii) of Theorem 3, the first inequality in (3.19) holds for $x \in (0, \pi/2)$.

In the cases of $0 < q \leq 34/35$ and $q \leq 0$, it can be proved in the same method, here we omit details of proof.

This completes the proof. \square

Letting $q = 1, 34/35$ in Theorem 4, we get

Corollary 6. (i) The double inequality

$$(3.20) \quad \left(1 - \frac{p_1}{3} + \frac{p_1}{3} \cos x\right)^{1/p_1} < \frac{\sin x}{x} < \left(1 - \frac{p_2}{3} + \frac{p_2}{3} \cos x\right)^{1/p_2}$$

holds for $x \in (0, \pi/2)$ if and only if $1.42034 \approx p_0 \leq p_1 \leq 3$ and $p_2 \leq 7/5$.

(ii) The double inequality

$$\left(1 - \frac{35p_1}{102} + \frac{35p_1}{102} \cos^{34/35} x\right)^{1/p_1} < \frac{\sin x}{x} < \left(1 - \frac{35p_2}{102} + \frac{35p_2}{102} \cos^{34/35} x\right)^{1/p_2}$$

holds for $x \in (0, \pi/2)$ if and only if $46/35 \leq p_1 \leq 102/35$ and $p_2 \leq p(34/35) = p_0^* \approx 1.27754$.

Remark 6. Letting $p_1 = 3/2, 2, 3$ and $p_2 = 7/5, 6/5, 1$ in (3.20) and using the decreasing property of $M(t; p, q)$ in p , we get

$$\begin{aligned} (\cos x)^{1/3} &< \left(\frac{1}{3} + \frac{2}{3} \cos x\right)^{1/2} < \left(\frac{1}{2} + \frac{1}{2} \cos x\right)^{2/3} < \frac{\sin x}{x} \\ &< \left(\frac{8}{15} + \frac{7}{15} \cos x\right)^{5/7} < \left(\frac{3}{5} + \frac{2}{5} \cos x\right)^{5/6} < \frac{2}{3} + \frac{1}{3} \cos x. \end{aligned}$$

Letting $q = 0$ in Theorem 4 we get

Corollary 7. For $x \in (0, \pi/2)$, the inequality

$$\frac{\sin x}{x} > \left(1 + \frac{p}{3} \ln(\cos x)\right)^{1/p}$$

holds if and only if $p \in [-8/5, 0)$.

3.2. In the case of $p = kq$. Letting $p/q = k$. Then $E_{p,q} = \{(p, q) : p \leq 0 \text{ or } 0 < p \leq 3q\}$ is changed into

$$E_{kq,q} = \{(k, q) : q \leq 0, k \geq 0 \text{ or } q \geq 0, k \leq 3\},$$

while $M(t; p, q)$ can be expressed as

$$M(t; kq, q) = \begin{cases} \left(1 - \frac{k}{3} + \frac{k}{3}t^q\right)^{1/(kq)} & \text{if } kq \neq 0, (k, q) \in E_{kq,q}, \\ \exp \frac{t^q - 1}{3q} & \text{if } k = 0, q \neq 0, \\ t^{1/3} & \text{if } q = 0. \end{cases}$$

Remark 7. Similar to the monotonicity of $M(t; p, q)$, we claim that $M(t; kq, q)$ is decreasing (increasing) in q if $k > (<) 3$, and is decreasing (increasing) in k if $q > (<) 0$.

In fact, logarithmic differentiations gives

$$\begin{aligned} \frac{\partial \ln M}{\partial q} &= \frac{1}{q^2} \left(\frac{qt^q \ln t}{3 - k + kt^q} - \frac{1}{k} \ln \left(1 - \frac{k}{3} + \frac{k}{3}t^q\right) \right) := \frac{M_2(t; k, q)}{q^2}, \\ \frac{\partial M_2}{\partial q} &= \frac{t^q \ln^2 t}{(3 - k + kt^q)^2} q(3 - k), \end{aligned}$$

which means that M_2 is decreasing (increasing) in q on $(0, \infty)$ and increasing (decreasing) on $(-\infty, 0)$ if $k > (<) 3$. Hence we have $M_2(t; k, q) < (>) M_2(t; k, 0) = 0$ if $k > (<) 3$, and then, M is decreasing (increasing) in q for $k > (<) 3$.

Analogously, the monotonicity of $M(t; kq, q)$ with respect to k easily follows from the following relations:

$$\begin{aligned} \frac{\partial \ln M}{\partial k} &= \frac{1}{k^2} \left(\frac{k}{q} \frac{t^q - 1}{3 - k + kt^q} - \frac{1}{q} \ln \left(1 - \frac{k}{3} + \frac{k}{3}t^q\right) \right) := \frac{M_3(t; k, q)}{k^2}, \\ \frac{\partial M_3}{\partial k} &= -\frac{k}{q} \frac{(t^q - 1)^2}{(3 - k + kt^q)^2}. \end{aligned}$$

As a direct consequence of Corollary 2, we have

Theorem 5. Let $(k, q) \in E_{kq,q} = \{(k, q) : q \leq 0, k \geq 0 \text{ or } q \geq 0, k \leq 3\}$. Then

(i) when $k \in (3, \infty)$, the inequality

$$\frac{\sin x}{x} > \left(1 - \frac{k}{3} + \frac{k}{3} \cos^q x\right)^{1/(kq)}$$

holds for $x \in (0, \pi/2)$ if $8/(5(3-k)) \leq q \leq 0$;

(ii) when $k \in [(35\pi^2 - 140)/136, 3)$, the double inequality

$$(3.21) \quad \left(1 - \frac{k}{3} + \frac{k}{3} \cos^{q_1} x\right)^{1/(kq_1)} < \frac{\sin x}{x} < \left(1 - \frac{k}{3} + \frac{k}{3} \cos^{q_2} x\right)^{1/(kq_2)}$$

holds for $x \in (0, \pi/2)$ if $q_2 \geq 8/(5(3-k))$ and $q_1 \leq 1$;

(iii) when $k \in [\pi^2/4 - 1, (35\pi^2 - 140)/136)$, the double inequality (3.21) holds for $x \in (0, \pi/2)$ if $q_2 \geq 8/(5(3-k))$ and $q_1 \leq 34/35$ or $(\pi^2/4 - 1)/k \leq q_1 \leq 1$;

(iv) when $k \in [7/5, \pi^2/4 - 1)$, the double inequality (3.21) holds for $x \in (0, \pi/2)$ if $q_2 \geq 8/(5(3-k))$ and $q_1 \leq 34/35$;

(v) when $k \in [23/17, 7/5)$, the double inequality (3.21) holds for $x \in (0, \pi/2)$ if $q_2 \geq 1$ and $q_1 \leq 34/35$;

(vi) when $k \in [0, 23/17)$, the double inequality (3.21) holds for $x \in (0, \pi/2)$ if $q_2 \geq 1$ and $q_1 \leq 8/(5(3-k))$;

(vii) when $k \in (-\infty, 0)$, the double inequality (3.21) holds for $x \in (0, \pi/2)$ if $q_2 \geq 1$ and $0 \leq q_1 \leq 8/(5(3-k))$.

Now we give sharp bounds $M(\cos x; kq, q)$ for $(\sin x)/x$ in the case of $k \in (0, 3)$. To this end, we note that $D_{kq,q}(\pi/2^-)$ can be expressed as

$$D_{kq,q}(\pi/2^-) = \begin{cases} \frac{1 - (\frac{2}{\pi})^{kq}}{3q} - \frac{1}{3q} & \text{if } q > 0, \\ -\infty & \text{if } q \leq 0. \end{cases}$$

It is easy to verify that there is a unique number $q(k) = \frac{\ln(1-k/3)}{k \ln(2/\pi)}$ such that $D_{kq,q}(\pi/2^-) > 0$ for $q > q(k)$ and $D_{kq,q}(\pi/2^-) < 0$ for $q < q(k)$. Also, we easily see that $k \mapsto q(k)$ is increasing on $(0, 3)$, and $q(p_0) = 1$, $q(p_0^*) = 34/35$. Meanwhile, $p = kq \leq (\geq) 3q - 8/5$ implies that $q \geq (\leq) \frac{8}{5(3-k)}$. Based on these preparations above, using the same method of proof as Theorem 3's, we can show the following theorem, whose proof is omitted.

Theorem 6. Let $k \in (0, 3)$ and $x \in (0, \pi/2)$. Then

(i) if $k \in [7/5, 3)$, then the inequality

$$(3.22) \quad \frac{\sin x}{x} < \left(1 - \frac{k}{3} + \frac{k}{3} \cos^q x\right)^{1/(kq)}$$

holds if and only if $q \geq 8/(5(3-k))$;

(ii) if $k \in (p_0, 3)$, then the inequality

$$(3.23) \quad \left(1 - \frac{k}{3} + \frac{k}{3} \cos^q x\right)^{1/(kq)} < \frac{\sin x}{x}$$

holds if and only if $q \leq (\ln(1-k/3))/(k \ln(2/\pi))$, where $p_0 \approx 1.42034$ is defined by (3.15);

(iii) if $k \in (0, 23/17]$, then the inequality (3.23) holds if and only if $q \leq 8/(5(3-k))$;

(iv) if $k \in (0, p_0^*)$, then the inequality (3.22) holds if and only if $q \geq (\ln(1-k/3))/(k \ln(2/\pi))$, where $p_0^* \approx 1.27754$ is defined by (3.15).

Taking $k = 1, 3/2, 2$ in Theorem 5, we get immediately

Corollary 8. For $x \in (0, \pi/2)$, (i) the double inequality

$$(3.24) \quad \left(\frac{2}{3} + \frac{1}{3} \cos^{q_1} x\right)^{1/q_1} < \frac{\sin x}{x} < \left(\frac{2}{3} + \frac{1}{3} \cos^{q_2} x\right)^{1/q_2}$$

holds if and only if $q_1 \leq 4/5$ and $q_2 \geq (\ln 3 - \ln 2) / (\ln \pi - \ln 2) \approx 0.89788$;
(ii) the double inequality

$$(3.25) \quad \left(\frac{1}{2} + \frac{1}{2} \cos^{q_1} x \right)^{2/(3q_1)} < \frac{\sin x}{x} < \left(\frac{1}{2} + \frac{1}{2} \cos^{q_2} x \right)^{2/(3q_2)}$$

holds if and only if $q_1 \leq (2 \ln 2) / (3 (\ln \pi - \ln 2)) \approx 1.0233$ and $q_2 \geq 16/15 \approx 1.0667$;

(iii) the double inequality

$$(3.26) \quad \left(\frac{1}{3} + \frac{2}{3} \cos^{q_1} x \right)^{1/(2q_1)} < \frac{\sin x}{x} < \left(\frac{1}{3} + \frac{2}{3} \cos^{q_2} x \right)^{1/(2q_2)}$$

holds if and only if $q_1 \leq (\ln 3) / (2 (\ln \pi - \ln 2)) \approx 1.2164$ and $q_2 \geq 8/5$.

Remark 8. Inequalities (3.24) is exactly (1.8) given in [17].

3.3. In the case of $p = 3q - 8/5$. When $p = 3q - 8/5$, $E_{p,q} = \{(p, q) : p \leq 0 \text{ or } 0 < p \leq 3q\}$ is changed into

$$(3.27) \quad E_{3q-8/5,q} = \{3q - 8/5 \leq 0 \text{ or } 0 < 3q - 8/5 \leq 3q\} = \mathbb{R}.$$

While $M(t; 3q - 8/5, q)$ can be completely written as

$$(3.28) \quad M(t; 3q - \frac{8}{5}, q) = \begin{cases} \left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right) t^q \right)^{5/(15q-8)} & \text{if } q \neq 0, \frac{8}{15}, \\ \left(1 - \frac{8}{15} \ln t\right)^{-5/8} & \text{if } q = 0, \\ \exp \frac{5(t^{8/15} - 1)}{8} & \text{if } q = \frac{8}{15}, \end{cases}$$

where $t = \cos x \in (0, 1)$ for $x \in (0, \pi/2)$. $N(t; 3q - 8/5, q)$ can be expressed as

$$\begin{aligned} N(t; 3q - \frac{8}{5}, q) &= \left(\left(\frac{2}{\pi}\right)^{3q-8/5} + \left(1 - \left(\frac{2}{\pi}\right)^{3q-8/5}\right) t^q \right)^{1/(3q-8/5)} \text{ if } q \neq \frac{8}{15}, q > 0 \\ N(t; 0, \frac{8}{15}) &= \left(\frac{2}{\pi}\right)^{1-t^{8/15}} \text{ if } q = \frac{8}{15}. \end{aligned}$$

For the monotonicity of $M(t; 3q - 8/5, q)$ in q , we can prove the following

Lemma 9. Let $(t, q) \mapsto M(t; 3q - 8/5, q)$ be defined on $(0, 1) \times \mathbb{R}$ by (3.28). Then $q \mapsto M(t; 3q - 8/5, q)$ is increasing on \mathbb{R} , and we have

$$(3.29) \quad \lim_{q \rightarrow -\infty} M(t; 3q - 8/5, q) = t^{1/3} \quad \text{and} \quad \lim_{q \rightarrow \infty} M(t; 3q - 8/5, q) = 1.$$

Proof. For $q \neq 0, 8/15$, logarithmic differentiation yields

$$\begin{aligned} \frac{\partial \ln M}{\partial q} &= -3 \frac{\ln \left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right) t^q \right)}{\left(3q - \frac{8}{5}\right)^2} - \frac{\frac{8}{15q^2} (1 - t^q) + t^q \left(\frac{8}{15q} - 1\right) \ln t}{\left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right) t^q\right) \left(3q - \frac{8}{5}\right)}, \\ \frac{\partial \ln M}{\partial t} &= 5q \frac{t^{q-1}}{(15q - 8) t^q + 8}, \\ \frac{\partial^2 \ln M}{\partial q \partial t} &= 40t^q \frac{\ln t^q - t^q + 1}{t((15q - 8) t^q + 8)^2} < 0, \end{aligned}$$

where the inequality holds due to $\ln x \leq x - 1$ for $x > 0$. Hence, $\partial(\ln M)/\partial q$ is decreasing in t , and so we have

$$\frac{\partial \ln M}{\partial q}(t; 3q - 8/5, q) > \frac{\partial \ln M}{\partial q}(1; 3q - 8/5, q) = 0,$$

which means that $q \mapsto M(t; 3q - 8/5, q)$ has increasing property. A direct computation yields (3.29). \square

By Corollary 1 we immediately get

Theorem 7. *Let $x \in (0, \pi/2)$.*

(i) If $q \geq 1$, then the double inequality

$$(3.30) \quad N(\cos x, 3q - \frac{8}{5}, q) < \frac{\sin x}{x} < M(\cos x, 3q - \frac{8}{5}, q),$$

hold, where $M(t; 3q - 8/5, q)$ and $N(t; 3q - 8/5, q)$ are defined by (3.6) and (3.7), respectively.

(ii) If $0 < q \leq 34/35$, then the double inequality (3.30) is reversed.

(iii) If $q \leq 0$, then the inequality

$$\frac{\sin x}{x} > M(\cos x, 3q - \frac{8}{5}, q)$$

holds.

In order to establish sharp inequalities $(\sin x)/x < (>) M(\cos x; 3q - 8/5, q)$, the following lemma is also needed.

Lemma 10. *Let $M(t; 3q - 8/5, q)$ be defined by (3.28). Then there is a unique $q_0 \approx 0.989681$ such that $D_{3q-8/5,q}(\pi/2^-) > 0$ for $q > q_0$ and $D_{3q-8/5,q}(\pi/2^-) < 0$ for $q < q_0$.*

Proof. From (3.13) we have

$$(3.31) \quad D_{3q-8/5,q}(\pi/2^-) = \begin{cases} \frac{1 - (\frac{2}{\pi})^{3q-8/5}}{3q-8/5} - \frac{1}{3q} & \text{if } q > 0, q \neq 8/15, \\ -\ln \frac{2}{\pi} - \frac{5}{8} < 0 & \text{if } q = 8/15, \\ -\infty & \text{if } q \leq 0. \end{cases}$$

It is obvious that $D_{3q-8/5,q}(\pi/2^-) < 0$ for $q \leq 8/15$, and it remains to show that $D_{3q-8/5,q}(\pi/2^-) < 0$ for $q \in (8/15, q_0)$ and $D_{3q-8/5,q}(\pi/2^-) > 0$ for $q > q_0$. For $q > 8/15$, we have

$$(3q - 8/5) D_{3q-8/5,q}(\pi/2^-) = \frac{8}{15q} - \left(\frac{2}{\pi}\right)^{3q-8/5} = L\left(\frac{8}{15q}, \left(\frac{2}{\pi}\right)^{3q-8/5}\right) \times v(q),$$

where $L(x, y)$ is the logarithmic mean of positive x and y , and

$$v(q) = \ln \frac{8}{15q} - (3q - 8/5) \ln \frac{2}{\pi}.$$

Differentiation leads us to

$$v'(q) = \frac{3}{q} \left(q - \frac{1}{3 \ln(\pi/2)} \right) \ln \frac{\pi}{2},$$

which tell us that v is increasing on $q \geq (3 \ln(\pi/2))^{-1} \approx 0.73814$ and decreasing on $8/15 < q \leq (3 \ln(\pi/2))^{-1}$. Hence, we have $v(q) < v(8/15) = 0$ for $8/15 < q \leq (3 \ln(\pi/2))^{-1}$, but

$$v(1) = \ln \frac{8}{15} - \frac{7}{5} \ln \frac{2}{\pi} \approx 3.6071 \times 10^{-3} > 0,$$

and it follows that there is a unique number q_0 to satisfy $v(q_0) = 0$ such that $v(q) < 0$ for $q \in (8/15, q_0)$ and $v(q) > 0$ for $q \in (q_0, \infty)$. Solving the equation $v(q_0) = 0$ by mathematical computer software we find that $q_0 \approx 0.989681$.

This proves the lemma. \square

Theorem 8. *Let $x \in (0, \pi/2)$. Then the inequality $(\sin x)/x > M(\cos x; 3q - 8/5, q)$ holds if and only if $q \leq 34/35$, while its reverse holds if and only if $q \geq q_0 \approx 0.989681$, where q_0 is the unique root of equation $D_{3q-8/5,q}(\pi/2^-) = 0$ on \mathbb{R}_+ , here $D_{3q-8/5,q}(\pi/2^-)$ is defined by (3.31).*

Proof. Clearly, the inequality $(\sin x)/x < (>) M(\cos x; 3q - 8/5, q)$ is equivalent to $D_{3q-8/5,q}(x) := S_{3q-8/5}(x)/C_q(x) - 1/3 > (<) 0$ for $x \in (0, \pi/2)$, where $M(t; 3q - 8/5, q)$ is defined by (3.28).

(i) We first prove that the inequality $(\sin x)/x > M(\cos x; 3q - 8/5, q)$ if and only if $q \leq 34/35$. The necessity is due to $\lim_{x \rightarrow 0^+} x^{-4} D_{3q-8/5,q}(x) \leq 0$. By (3.12) we get

$$\lim_{x \rightarrow 0^+} \frac{D_{3q-8/5,q}(x)}{x^4} = \frac{1}{135} \left(q - \frac{34}{35} \right) \leq 0,$$

which yields $q \leq 34/35$. The sufficiency is obtained from Theorem 7.

(ii) Now we show that the inequality $(\sin x)/x < M(\cos x; 3q - 8/5, q)$ if and only if $q \geq q_0 \approx 0.989681$. The necessity can be derived by the relation $D_{3q-8/5,q}(\pi/2^-) = S_p(\pi/2^-)/C_q(\pi/2^-) - 1/3 \geq 0$, which follows by part (ii) of Lemma 10.

Next we show the sufficiency. In the case of $q \geq 1$, it is obviously true by Theorem 7. In the case of $q \in [p_0, 1)$, we see that $f_2(x)$ defined by (2.13) can be written as

$$(3.32) \quad f_2(x) = (3q - 8/5)A(x) - qB(x) + C(x) = -(B(x) - 3A(x))(q - g_2(x)),$$

where $g_2(x)$ is defined by (2.21) and $(B(x) - 3A(x)) > 0$ for $x \in (0, \pi/2)$. By Lemma 7, we see that $g_2 = (C(x) - (8/5)A(x))/(B(x) - 3A(x))$ is increasing on $(0, \pi/2)$, and so $x \mapsto q - g_2(x) := j(x)$ is decreasing on $(0, \pi/2)$. But,

$$j(0^+) = q - 34/35 > 0 \quad \text{and} \quad j\left(\frac{\pi^-}{2}\right) = q_0 - 1 < 0,$$

then it is seen that there is a unique number $x_3 \in (0, \pi/2)$ such that $j(x) > 0$ for $x \in (0, x_3)$ and $j(x) < 0$ for $x \in (x_3, \pi/2)$. This together with (3.32) and $(B(x) - 3A(x)) > 0$ means that $f_2(x) < 0$ for $x \in (0, x_3)$ and $f_2(x) > 0$ for $x \in (x_3, \pi/2)$. From the relations (2.22) and (2.12) it is derived that the function $x \mapsto S'_{3q-8/5}(x)/C'_q(x)$ is increasing on $(0, x_3]$ and decreasing on $(x_3, \pi/2)$. Since $C'_q(x) = \cos^{q-1} x \sin x > 0$ for $x \in (0, \pi/2)$ and for $q \in [p_0, 1)$ the relation

$$\frac{S_{3q-8/5}\left(\frac{\pi^-}{2}\right) - S_{3q-8/5}(0^+)}{C_q\left(\frac{\pi^-}{2}\right) - C_q(0^+)} > \frac{1}{3}$$

holds, make use of Lemma 3 it is deduced that the inequality

$$\frac{S_{3q-8/5}(x) - S_{3q-8/5}(0^+)}{C_q(x) - C_q(0^+)} > \frac{1}{3}$$

holds for all $x \in (0, \pi/2)$, that is, $S_{3q-8/5}(x)/C_q(x) > 1/3$ is valid for $x \in (0, \pi/2)$ in the case of $q \in [p_0, 1)$. Thus the sufficiency follows from Lemma 3.

This completes the proof of this theorem. \square

We close this section by giving a very nice chain of inequalities for trigonometric functions. Taking $q = -\infty, 0, 8/15, 7/10, 4/5, 13/15, 34/35; q_0, 1, 6/5$, we deduce that

Corollary 9. For $x \in (0, \pi/2)$, the chain of inequalities holds:

$$\begin{aligned}
\cos^{1/3} x &< \cdots < \left(1 - \frac{8}{15} \ln(\cos x)\right)^{-5/8} < \left(\frac{8}{3} - \frac{5}{3} \cos^{1/5} x\right)^{-1} \\
&< \exp\left(\frac{5}{8} \cos^{8/15} x - \frac{5}{8}\right) < \left(\frac{5}{21} \cos^{7/10} x + \frac{16}{21}\right)^2 \\
&< \left(\frac{1}{3} \cos^{4/5} x + \frac{2}{3}\right)^{5/4} < \frac{5}{13} \cos^{13/15} x + \frac{8}{13} \\
&< \left(\frac{23}{51} \cos^{34/35} x + \frac{28}{51}\right)^{35/46} < \frac{\sin x}{x} < \left(\frac{7}{15} \cos x + \frac{8}{15}\right)^{5/7} \\
&< \sqrt{\frac{5}{9} \cos^{6/5} x + \frac{4}{9}} < \frac{2 + \cos x}{3}.
\end{aligned}$$

Proof. Employing Theorem 8 and increasing property of $M(\cos x; 3q - 8/5, q)$ with respect to q , we get the all inequalities except for the last one. In order for the last one to be valid, it suffices that the inequality

$$5 \cos^{6/5} x + 4 < (2 + \cos x)^2$$

holds for $x \in (0, \pi/2)$. With $\cos^{1/5} x = t$, then we get

$$5t^6 + 4 - (2 + t^5)^2 = -t^5(t-1)^2(t^3 + 2t^2 + 3t + 4) < 0,$$

which proves the desired inequality. \square

4. APPLICATIONS

As demonstrated by Zhu in [7], an inequality for trigonometric functions can be changed into another one by making identical transformations or changes of variables. Based on our results in previous sections, we can obtain corresponding inequalities in the same way. For example, multiplying the each sides in double inequality (3.1) by $((\sin x)/x)^{-q}$ yields a new Huygens type one:

$$\left(\frac{2}{\pi}\right)^p \left(\frac{x}{\sin x}\right)^q + \left(1 - \left(\frac{2}{\pi}\right)^p\right) \left(\frac{x}{\tan x}\right)^q < \left(\frac{\sin x}{x}\right)^{p-q} < \left(1 - \frac{p}{3q}\right) \left(\frac{x}{\sin x}\right)^q + \frac{p}{3q} \left(\frac{x}{\tan x}\right)^q$$

if $q \geq 1$ and $0 < p \leq 3q - 8/5$. And then, Theorem 1 can be restated in Huygens type.

Next we will give some other applications.

4.1. Shafer-Fink type and Carlson type inequalities. In [1, 3, p. 247, 3.4.31], it was listed that the inequality

$$\arcsin x > \frac{6(\sqrt{x+1} - \sqrt{1-x})}{4 + \sqrt{x+1} + \sqrt{1-x}} > \frac{3x}{2 + \sqrt{1-x^2}}$$

hold for $x \in (0, 1)$, which is due to Shafer [24]. Fink [25] proved that the double inequality

$$\frac{3x}{2 + \sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2 + \sqrt{1-x^2}}$$

is true for $x \in [0, 1]$. There has some improvements, generalizations of Shafer-Fink inequality (see [26], [27], [28], [29], [30], [31]).

Carlson [32, (1.14)] inequality states that the double inequality

$$(4.1) \quad \frac{6(1-t)^{1/2}}{2\sqrt{2} + (1+t)^{1/2}} < \arccos t < \frac{2^{2/3}(1-t)^{1/2}}{(1+t)^{1/6}}$$

holds for $t \in (0, 1)$. As a corollary of Theorem Zhu given in [7, Theorem 5], he gave a generalization of (4.1).

Shafer-Fink type and Carlson inequalities are essentially attributed to ones involving the functions $(\sin x)/x$ and $\cos x$, where $x \in (0, \pi/2)$. Therefore, after making a change of variable $\sin x = t$ or $\cos x = t$, an inequality for trigonometric functions may be changed into a Shafer-Fink type or Carlson type one. For example, by letting $\sin x = t$, the double inequality (3.5) can be changed into

$$(4.2) \quad \frac{t}{M(\sqrt{1-t^2}; p, q)} < \arcsin t < \frac{t}{N(\sqrt{1-t^2}; p, q)}$$

if $q \geq 1$ and $p \leq 3q - 8/5$ and $(p, q) \in E_{p,q}$, where M and N are defined by (3.6) and (3.7), respectively. Thus Theorem 1 can be restated as follows.

Proposition 4 (Shafer-Fink type inequalities). *Let $t \in (0, 1)$ and $(p, q) \in E_{p,q}$.*

(i) *If $q \geq 1$ and $p \leq 3q - 8/5$, then the double inequality (4.2) holds.*

(ii) *If $34/35 < q \leq 1$ and $p \geq \pi^2/4 - 1$, then the double inequalities (4.2) is reversed.*

(iii) *If $0 < q \leq 34/35$ and $p \geq 3q - 8/5$, then all the double inequalities (4.2) is reversed.*

(iv) *If $q \leq 0$ and $p \geq 3q - 8/5$, then the inequality*

$$\arcsin t < \frac{t}{M(\sqrt{1-t^2}; p, q)}$$

holds.

Clearly, Proposition 4 is also true if replacing $\arcsin t$, t and $\sqrt{1-t^2}$ with $\arccos t$, $\sqrt{1-t^2}$ and t , respectively.

Next we give more refined Shafer-Fink type and Carlson type inequalities. Employing the monotonicity of $x \mapsto T_{p,q}(x/2)$ on $(0, \pi/2)$ given in Proposition 1, we get for $pq \neq 0$,

$$\frac{1}{3} < T_{p,q}\left(\frac{\pi}{2}\right) = \frac{S_p(x/2)}{C_q(x/2)} < \frac{S_p(\pi/4)}{C_q(\pi/4)} = T_{p,q}\left(\frac{\pi}{4}\right)$$

if $q \geq 1$ and $p \leq 3q - 8/5$, that is,

$$\begin{aligned} \frac{1}{3} &< \frac{q}{p} \frac{1 - \left(\frac{2\sin(x/2)}{x}\right)^p}{1 - \cos^q(x/2)} < \frac{q}{p} \frac{1 - (2\sqrt{2}/\pi)^p}{1 - 2^{-q/2}} = \frac{1}{c_{p,q}} \text{ if } p \neq 0, \\ \frac{1}{3} &< q \frac{-\ln \frac{2\sin(x/2)}{x}}{1 - \cos^q(x/2)} < q \frac{-\ln(2\sqrt{2}/\pi)}{1 - 2^{-q/2}} = \frac{1}{c_{0,q}} \text{ if } p = 0. \end{aligned}$$

Also, it is easy to check that the largest set of $\{(p, q)\}$ over the real numbers field such that both the inequalities

$$\begin{aligned} 1 - \frac{p}{qc_{p,q}} + \frac{p}{qc_{p,q}} \cos^q(x/2) &> 0 \text{ if } q \neq 0, \\ 1 + \frac{p}{c_{p,0}} \ln(\cos \frac{x}{2}) &> 0 \text{ if } q = 0 \end{aligned}$$

hold for all $x \in (0, \pi/2)$ is \mathbb{R} . While the largest set of $\{(p, q)\}$ such that both the inequalities

$$\begin{aligned} 1 - \frac{p}{3q} + \frac{p}{3q} \cos^q(x/2) &> 0 \text{ if } q \neq 0, \\ 1 + \frac{p}{3} \ln(\cos \frac{x}{2}) &> 0 \text{ if } q = 0 \end{aligned}$$

hold for all $x \in (0, \pi/2)$ is $\{(p, q) : p < 3q/(1 - 2^{-q/2}) \text{ if } q \neq 0 \text{ and } p < 6/\ln 2 \text{ if } q = 0\}$.

Now let $x = \arcsin t$. Then

$$\sin \frac{x}{2} = \frac{\sqrt{1+t} - \sqrt{1-t}}{2}, \quad \cos \frac{x}{2} = \frac{\sqrt{1+t} + \sqrt{1-t}}{2}.$$

If let $x = \arccos t$. Then

$$\sin \frac{x}{2} = \frac{\sqrt{1-t}}{\sqrt{2}}, \quad \cos \frac{x}{2} = \frac{\sqrt{1+t}}{\sqrt{2}}.$$

And then, by Proposition 1 we get respectively

Proposition 5 (Shafer-Fink type inequalities). *Let $t \in (0, 1)$ and $(p, q) \in \{(p, q) : p < 3q/(1 - 2^{-q/2}) \text{ if } q \neq 0 \text{ and } p < 6/\ln 2 \text{ if } q = 0\}$. Then*

(i) when $q \geq 1$ and $p \leq 3q - 8/5$, the double inequalities

$$(4.3) \quad \frac{(\sqrt{1+t} - \sqrt{1-t})^p}{1 - \frac{p}{3q} + \frac{p}{3q} 2^{-q} (\sqrt{1+t} + \sqrt{1-t})^q} < \arcsin^p t < \frac{(\sqrt{1+t} - \sqrt{1-t})^p}{1 - \frac{p}{qc_{p,q}} + \frac{p}{qc_{p,q}} 2^{-q} (\sqrt{1+t} + \sqrt{1-t})^q} \text{ if } p > 0,$$

$$(4.4) \quad (\sqrt{1+t} - \sqrt{1-t}) \exp \frac{1-2^{-q}(\sqrt{1+t} + \sqrt{1-t})^q}{3q} < \arcsin t < (\sqrt{1+t} - \sqrt{1-t}) \exp \frac{1-2^{-q}(\sqrt{1+t} + \sqrt{1-t})^q}{qc_{0,q}} \text{ if } p = 0,$$

$$(4.5) \quad \frac{(\sqrt{1+t} - \sqrt{1-t})^p}{1 - \frac{p}{qc_{p,q}} + \frac{p}{qc_{p,q}} 2^{-q} (\sqrt{1+t} + \sqrt{1-t})^q} < \arcsin^p t < \frac{(\sqrt{1+t} - \sqrt{1-t})^p}{1 - \frac{p}{3q} + \frac{p}{3q} 2^{-q} (\sqrt{1+t} + \sqrt{1-t})^q} \text{ if } p < 0$$

hold, where 3 and

$$(4.6) \quad c_{p,q} = \begin{cases} \frac{p}{q} \frac{1-2^{-q/2}}{1-(2\sqrt{2}/\pi)^p} & \text{if } pq \neq 0, \\ \frac{1}{q} \frac{1-2^{-q/2}}{\ln(\pi/2\sqrt{2})} & \text{if } p = 0, q \neq 0, \\ \frac{p}{2} \frac{\ln 2}{1-(2\sqrt{2}/\pi)^p} & \text{if } p \neq 0, q = 0, \\ \frac{1}{2} \frac{\ln 2}{\ln(\pi/2\sqrt{2})} & \text{if } p = q = 0 \end{cases}$$

are the best constants;

(ii) when $34/35 < q \leq 1$ and $\pi^2/4 - 1 \leq p < 3q/(1 - 2^{-q/2})$, the double inequality (4.3) is reversed;

(iv) when $q \leq 34/35$ and $3q - 8/5 \leq p < 3q/(1 - 2^{-q/2})$, all the double inequalities (4.3), (4.4) and (4.5) are reversed.

Proposition 6 (Carlson type inequalities). *Let $t \in (0, 1)$ and $(p, q) \in \{(p, q) : p < 3q/(1 - 2^{-q/2}) \text{ if } q \neq 0 \text{ and } p < 6/\ln 2 \text{ if } q = 0\}$. Then*

(i) when $q \geq 1$ and $p \leq 3q - 8/5$, the double inequalities

$$(4.7) \quad \frac{(2\sqrt{1-t})^p}{1 - \frac{p}{3q} + \frac{p}{3q} 2^{-q/2} (\sqrt{1+t})^q} < \arccos^p t < \frac{(2\sqrt{1-t})^p}{1 - \frac{p}{qc_{p,q}} + \frac{p}{qc_{p,q}} 2^{-q/2} (\sqrt{1+t})^q} \text{ if } p > 0,$$

$$(4.8) \quad \sqrt{2}\sqrt{1-t} \exp \frac{1-2^{-q/2}(\sqrt{1+t})^q}{3q} \\ < \arccos t < \sqrt{2}\sqrt{1-t} \exp \frac{1-2^{-q/2}(\sqrt{1+t})^q}{qc_{0,q}} \text{ if } p = 0,$$

$$(4.9) \quad \frac{(2\sqrt{1-t})^p}{1-\frac{p}{qc_{p,q}}+\frac{p}{qc_{p,q}}2^{-q/2}(\sqrt{1+t})^q} < \arccos^p t < \frac{(2\sqrt{1-t})^p}{1-\frac{p}{3q}+\frac{p}{3q}2^{-q/2}(\sqrt{1+t})^q} \text{ if } p < 0$$

hold, where 3 and $c_{p,q}$ defined by (4.6) are the best constants;

(ii) when $34/35 < q \leq 1$ and $\pi^2/4 - 1 \leq p < 3q/(1-2^{-q/2})$, the double inequality (4.7) is reversed;

(iv) when $q \leq 34/35$ and $3q - 8/5 \leq p < 3q/(1-2^{-q/2})$, all the double inequalities (4.7), (4.8) and (4.9) are reversed.

4.2. Inequalities for means. Let G, A, Q, P, T and U stand for the geometric, arithmetic, quadratic, the first Seiffert [33], the second Seiffert [34] and Yang's means [14] of distinct positive numbers a and b defined by

$$G = G(a, b) = \sqrt{ab}, \quad A = A(a, b) = \frac{a+b}{2}, \quad Q = Q(a, b) = \sqrt{\frac{a^2+b^2}{2}}, \\ P = P(a, b) = \frac{a-b}{2 \arcsin \frac{a-b}{a+b}}, \quad T = T(a, b) = \frac{a-b}{2 \arctan \frac{a-b}{a+b}}, \quad U = U(a, b) = \frac{a-b}{\sqrt{2} \arcsin \frac{a-b}{\sqrt{2ab}}},$$

respectively. The Schwab-Borchardt mean of two numbers $a \geq 0$ and $b > 0$, denoted by $SB(a, b)$, is defined as [35, Theorem 8.4], [36, (2.3)], [37]

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos(a/b)} & \text{if } a < b, \\ a & \text{if } a = b, \\ \frac{\sqrt{a^2-b^2}}{\operatorname{arccosh}(a/b)} & \text{if } a > b. \end{cases}$$

Very recently, Yang [14, Theorem 3.1] has defined a family of two-parameter trigonometric sine means as follows.

Definition 1. Let $b \geq a > 0$ with and $p, q \in [-2, 2]$ such that $0 \leq p+q \leq 3$, and let $\tilde{S}(p, q, t)$ be defined by

$$(4.10) \quad \tilde{S}(p, q, t) = \begin{cases} \left(\frac{q \sin pt}{p \sin qt} \right)^{1/(p-q)} & \text{if } pq(p-q) \neq 0, \\ \left(\frac{\sin pt}{pt} \right)^{1/p} & \text{if } q = 0, p \neq 0, \\ \left(\frac{\sin qt}{qt} \right)^{1/q} & \text{if } p = 0, q \neq 0, \\ e^{t \cot pt - 1/p} & \text{if } p = q \neq 0, \\ 1 & \text{if } p = q = 0. \end{cases}$$

Then $\mathcal{S}_{p,q}(a, b)$ defined by

$$(4.11) \quad \mathcal{S}_{p,q}(a, b) = b \times \tilde{S}(p, q, \arccos(a/b)) \text{ if } a \neq b \text{ and } \mathcal{S}_{p,q}(a, a) = a$$

is called a two-parameter sine mean of a and b .

As a special case, for $b \geq a > 0$,

$$\mathcal{S}_{1,0}(a, b) = b \frac{\sin t}{t} \Big|_{t=\arccos(a/b)}$$

is a mean of a and b . Clearly, $\mathcal{S}_{1,0}(a, b) = SB(a, b)$. Thus, after replacing t by $\arccos(a/b)$ and multiplying each sides of those inequalities showed in previous section by b , they can be written as corresponding inequalities for means. For example, Theorem 6 can be restated as follows.

Proposition 7. *Let $k \in (0, 3)$ and $b > a > 0$. Then*

(i) *if $k \in [7/5, 3)$, then the inequality*

$$(4.12) \quad SB(a, b) < b^{1-1/k} \left(\left(1 - \frac{k}{3} \right) b^q + \frac{k}{3} a^q \right)^{1/(kq)}$$

holds if and only if $q \geq 8/(5(3-k))$;

(ii) *if $k \in (p_0, 3)$, then the inequality*

$$(4.13) \quad b^{1-1/k} \left(\left(1 - \frac{k}{3} \right) b^q + \frac{k}{3} a^q \right)^{1/(kq)} < SB(a, b)$$

holds if and only if $q \leq (\ln(1-k/3))/(k \ln(2/\pi))$, where $p_0 \approx 1.42034$ is defined by (3.15);

(iii) *if $k \in (0, 23/17]$, then the inequality (4.13) holds if and only if $q \leq 8/(5(3-k))$;*

(iv) *if $k \in (0, p_0^*)$, then the inequality (4.12) holds if and only if $q \geq (\ln(1-k/3))/(k \ln(2/\pi))$, where $p_0^* \approx 1.27754$ is defined by (3.15).*

As another example, Theorem 8 can be restated in the following form.

Proposition 8. *Let $b > a > 0$. Then the inequality $SB(a, b) > b \times M(a/b; 3q - 8/5, q)$ holds if and only if $q \leq 34/35$, while its reverse holds if and only if $q \geq q_0 \approx 0.989681$, where q_0 is the unique root of equation $D_{3q-8/5, q}(\pi/2^-) = 0$ on \mathbb{R}_+ , here $D_{3q-8/5, q}(\pi/2^-)$ is defined by (3.31).*

Further, let $m = m(a, b)$ and $M = M(a, b)$ be two means of a and b with $m(a, b) < M(a, b)$ for all $a, b > 0$. Clearly, making a change of variables $a \rightarrow m(a, b)$ and $b \rightarrow M(a, b)$, $\mathcal{S}_{p,q}(m, M)$ is still a mean of a and b which lie in m and M . Particularly, taking $(m, M) = (G, A)$, (A, Q) , (G, Q) , respectively, we can obtain new symmetric means as follows:

$$\begin{aligned} \mathcal{S}_{1,0}(G, A) &= SB(G, A) = \frac{a-b}{2 \arcsin \frac{a-b}{a+b}} = P(a, b), \\ \mathcal{S}_{1,0}(A, Q) &= SB(A, Q) = \frac{a-b}{2 \arctan \frac{a-b}{a+b}} = T(a, b), \\ \mathcal{S}_{1,0}(G, Q) &= SB(G, Q) = \frac{a-b}{\sqrt{2} \arcsin \frac{a-b}{\sqrt{2ab}}} = U(a, b), \end{aligned}$$

where P, T, U are the first Seiffert, the second Seiffert, Yang's means.

And then, after letting $t = \arccos(a/b)$ and multiplying each sides of those inequalities showed in previous section by b , and replacing $(a, b, SB(a, b))$ with (G, A, P) , (A, Q, T) , (G, Q, U) , we can establish corresponding inequalities for symmetric means. For example, Theorem 8 can be restated in the following form.

Proposition 9. Let $a, b > 0$ with $a \neq b$. Then the inequality

$$P > \begin{cases} A^{(10q-8)/(15q-8)} \left(\frac{8}{15q} A^q + \left(1 - \frac{8}{15q} \right) G^q \right)^{5/(15q-8)} & \text{if } q \neq 0, 8/15, \\ A \left(1 - \frac{8}{15} \ln \frac{G}{A} \right)^{-5/8} & \text{if } q = 0, \\ A \exp \frac{5((G/A)^{8/15} - 1)}{8} & \text{if } q = 8/15 \end{cases}$$

hold if and only if $q \leq 34/35$, while their reverse hold if and only if $q \geq q_0 \approx 0.989681$, where q_0 is the unique root of equation $D_{3q-8/5,q}(\pi/2^-) = 0$ on \mathbb{R}_+ , here $D_{3q-8/5,q}(\pi/2^-)$ is defined by (3.31).

Remark 9. When replacing (G, A, P) by (A, Q, T) and (G, Q, U) , the above proposition are still valid.

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